Polynomial interpolation on interlacing rectangular grids

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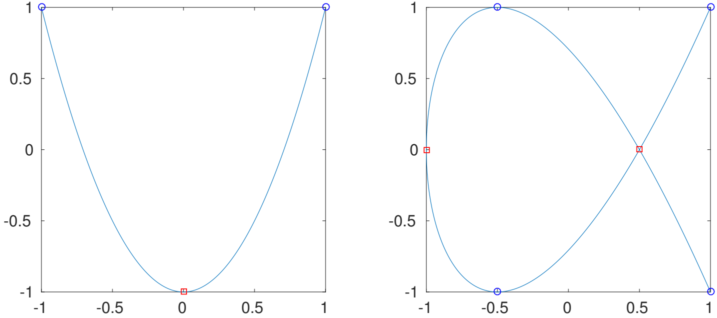
Padua points

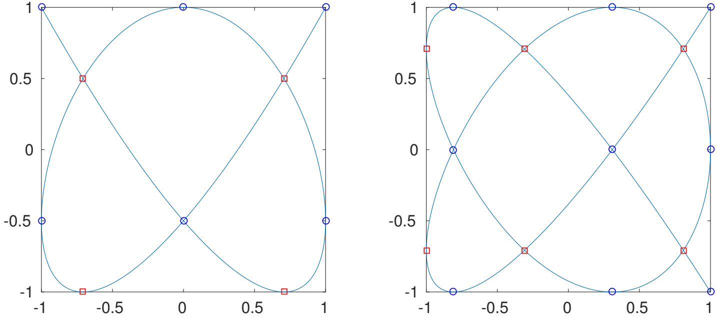
The intersections of the Lissajous curve $\gamma(t) = (\cos nt, \cos(n+1)t),\ t \in [0, \pi]$, with itself and the boundary of $[-1, 1]^2$.

Caliari, De Marchi, and M. Vianello (2005). 'Bivariate Lagrange interpolation on the square at new nodal sets'.

Examples, $n = 1$, $n = 2$



Examples, $n = 3$, $n = 4$



Properties

- Two interlacing rectangular grids.
- Unisolvent for polynomial interpolation of degree n.
- The Lebesgue constant grows with minimal order $O(\log^2(n))$.
- The associated cubature rule has degree of precision $2n - 1$ with respect to the Chebyshev weighting.

Bos, Caliari, and De Marchi, Vianello, and Xu (2006). Bivariate Lagrange interpolation at the Padua points: the generating curve approach.

The cubature rule is

$$\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} f(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}\,dx\,dy \approx \sum_{A\in\mathrm{Pad}_n} w_A f(A),$$

with weights

$$w_A = \frac{1}{n(n+1)}\begin{cases} 1/2 & \text{if } A \text{ is a vertex point;} \\ 1 & \text{if } A \text{ is an edge point;} \\ 2 & \text{if } A \text{ is an interior point.} \end{cases}$$

This rule is exact for any polynomial p of degree $\leq 2n-1$: one can show that

$$\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} p(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}\,dx\,dy = \frac{1}{\pi}\int_{0}^{\pi} q(t)\,dt,$$

where

$$q(t) = p(\cos nt, \cos(n+1)t),$$

and one can then apply the composite trapezoidal rule to q.

Other point sets

Points sets with similar interpolation and cubature properties were studied earlier in
Morrow and Patterson (1978), Construction of algebraic cubature rules using polynomial ideal theory,
and
Yuan Xu (1996), Lagrange interpolation on Chebyshev points of two variables.
In general, the associated polynomial space does not have total degree, but has the form

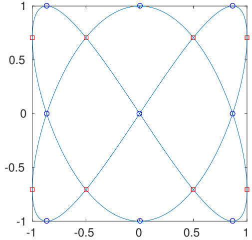
$$\Pi(L) = \mathrm{span}\{x^i y^j : (i,j) \in L\}$$

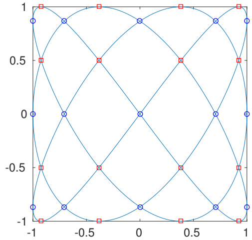
for some lower set $L \subset \mathbb{N}_0^2$, meaning that if $(k,l) \in L$ and $(i,j) \in \mathbb{N}_0^2$ and $(i,j) \le (k,l)$ then $(i,j) \in L$.

More recently, in

Erb, Kaethner, Ahlborg, and Buzug (2016), Bivariate Lagrange interpolation at the node points of non-degenerate Lissajous curves.

various families of point sets generated by Lissajous curves have been shown to have similar properties to the Padua points.





Changing the spacing of the points?

If we change the spacing of the Padua points, are they still unisolvent? This question was studied in

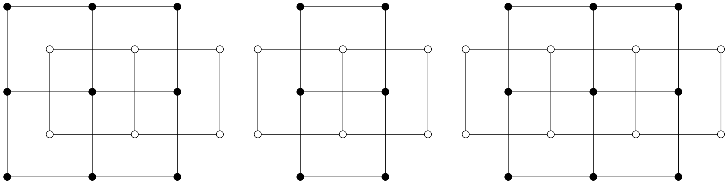
Bos, De Marchi, Waldron (2009). On the Vandermonde determinant of Padua-like points.
De Marchi and K. Usevich (2014), On certain multivariate Vandermonde determinants whose variables separate.
Pierro de Carmago and De Marchi (2015). A few remarks on "On certain Vandermonde determinants whose variables separate".

New approach

Defining $B_{k,l} = \{(i,j) : 0 \le i \le k,\ 0 \le j \le l\}$, the point set is the union of any two interlaced rectangular grids in $\mathbb{R}^2$,

$$U = \{(u_i, v_j) : (i,j) \in B_{\mu,\nu}\}, \qquad X = \{(x_i, y_j) : (i,j) \in B_{m,n}\}.$$

Interlacing implies that $|m - \mu| \le 1$ and $|n - \nu| \le 1$. We will assume without loss of generality that $n \le \nu$.

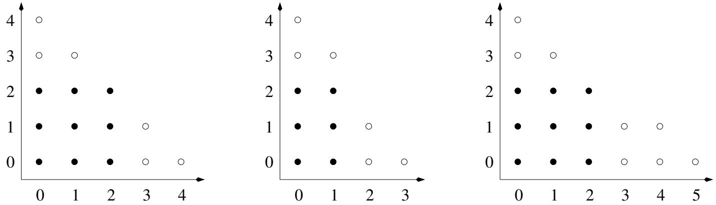


Choose an associated polynomial space

Let $K_1 \subset B_{m,n}$ be any lower set, let

$$K_2 = \{(i,j) : (m-i, n-j) \in B_{m,n} \setminus K_1\}, \quad \text{and}$$
$$L = B_{\mu,\nu} \cup (K_1 + (\mu+1, 0)) \cup (K_2 + (0, \nu+1)).$$

Then L is also a lower set (in the special case that $m = \mu + 1$ we must include the index pair $(0, n)$ in K_1).



Theorem

Given the values of a real function f on $U \cup X$, there is a unique polynomial $p \in \Pi(L)$ such that $p = f$ on $U \cup X$.

Tensor-product interpolation

Let us start by using tensor-product interpolation on U. Letting

$$a(x) = (x - u_0)(x - u_1)\dots(x - u_\mu),$$
$$b(y) = (y - v_0)(y - v_1)\dots(y - v_\nu),$$

we can express any $p \in \Pi(L)$ uniquely as

$$p(x,y) = p_0(x,y) + a(x)p_1(x,y) + b(y)p_2(x,y),$$

where $p_0 \in \Pi(B_{\mu,\nu})$, $p_1 \in \Pi(K_1)$, and $p_2 \in \Pi(K_2)$. So let p_0 be the tensor-product interpolant to f on U, and we will have $p = f$ on $U \cup X$ if we can find $p_1 \in \Pi(K_1)$ and $p_2 \in \Pi(K_2)$ such that

$$a(x_i)p_1(x_i,y_j) + b(y_j)p_2(x_i,y_j) = g_{ij}, \quad (i,j) \in B_{m,n},$$

where

$$g_{ij} := f(x_i,y_j) - p_0(x_i,y_j).$$

Newton form

To solve the equations let

$$p_1(x,y) = \sum_{(k,l)\in K_1} c_{kl}\phi_k(x)\psi_l(y),$$
$$p_2(x,y) = \sum_{(k,l)\in K_2} d_{kl}\phi_k(x)\psi_l(y),$$

for coefficients $c_{kl}, d_{kl} \in \mathbb{R}$, where $\phi_0(x) = \psi_0(y) = 1$, and

$$\phi_k(x) = (x - x_0)(x - x_1)\cdots(x - x_{k-1}), \quad k \geq 1,$$

$$\psi_l(y) = (y - y_0)(y - y_1)\cdots(y - y_{l-1}), \quad l \geq 1.$$

Then take divided differences

Applying the divided difference operator $[x_0, \ldots, x_i; y_0, \ldots, y_j]$ to each side of the equation, and simplying using the Leibniz rule, gives

$$\sum_{k:(k,j)\in K_1} a_{ki} c_{kj} + \sum_{l:(i,l)\in K_2} b_{lj} d_{il} = h_{ij}, \qquad (i,j) \in B_{m,n},$$

where

$$a_{ki} := \begin{cases} [x_k, \ldots, x_i]a, & k \le i; \\ 0, & k > i, \end{cases} \qquad b_{lj} := \begin{cases} [y_l, \ldots, y_j]b, & l \le j; \\ 0, & l > j, \end{cases}$$

and

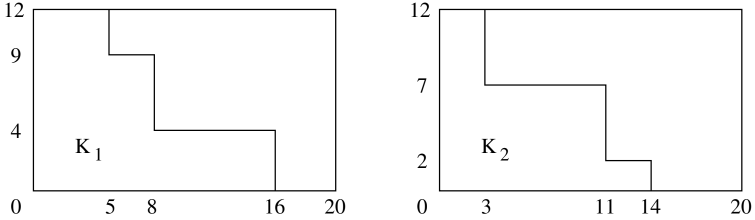
$$h_{ij} = [x_0, \ldots, x_i; y_0, \ldots, y_j](f - p_0).$$

Solution

Let (i_r, j_r), $r = 1, 2, \dots, s$, be the maximal points of K_1,

$$0 \le i_1 < \cdots < i_s \le m, \qquad n \ge j_1 > \cdots > j_s \ge 0.$$

Assume that $(0, n) \in K_1$ and $(m, 0) \notin K_1$ (the remaining cases are similar). Then $j_1 = n$ and $i_s < m$ and K_2 has maximal points $(m - i_r - 1, n - j_{r+1} - 1)$, $r = 1, \dots, s-1$, and $(m - i_s - 1, n)$.



For each $r = 1, 2, \ldots, s$ apply two steps.

(i) For $j = j_{r+1} + 1, \ldots, j_r$, solve

$$\sum_{k=0}^{i_r} a_{ki} c_{kj} = h_{ij} - \sum_{l=0}^{n-j_\alpha-1} b_{lj} d_{il},$$

$$\alpha = 1, \ldots, r, \quad i = m - i_\alpha, \ldots, m - i_{\alpha-1} - 1,$$

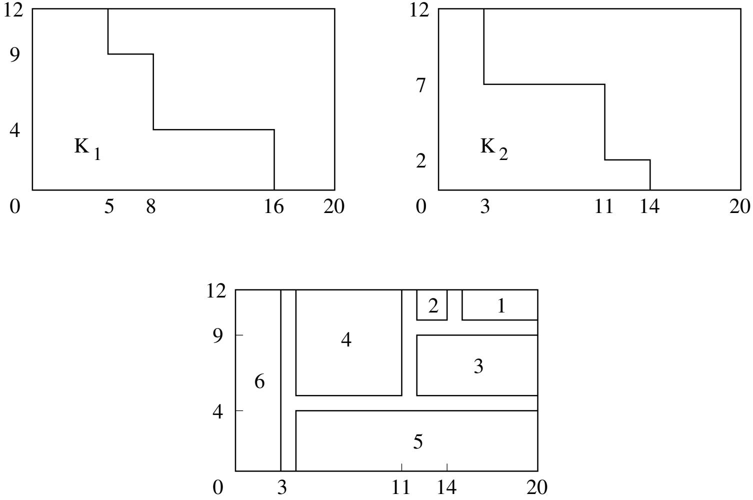
for the unknowns $c_{0,j}, \ldots, c_{i_r,j}$.

(ii) For $i = m - i_{r+1}, \ldots, m - i_r - 1$, solve

$$\sum_{l=0}^{n-j_{r+1}-1} b_{lj} d_{il} = h_{ij} - \sum_{k=0}^{i_\alpha} a_{ki} c_{kj},$$

$$\alpha = 1, \ldots, r, \quad j = j_{\alpha+1} + 1, \ldots, j_\alpha,$$

for the unknowns $d_{i,0}, \ldots, d_{i,n-j_{r+1}-1}$.



Odd-numbered blocks find rows of c-coefficients, even-numbered blocks find columns of d-coefficients.

For general m, consider the divided difference matrix

$$A = \begin{bmatrix} a_{0,m} & \cdots & a_{m,m} \\ \vdots & & \vdots \\ a_{0,0} & \cdots & a_{m,0} \end{bmatrix} = \begin{bmatrix} [x_0, \ldots, x_m]a & \cdots & [x_{m-1}, x_m]a & [x_m]a \\ [x_0, \ldots, x_{m-1}]a & \cdots & [x_{m-1}]a & 0 \\ \vdots & \iddots & \iddots & \vdots \\ [x_0]a & 0 & \cdots & 0 \end{bmatrix}.$$

Let A_r be the upper left submatrix of A of dimension $r+1$, $r = 0, 1, \ldots, m$.
For any choice of the lower set K_1, the linear system is solvable if all these submatrices are non-singular (and the same for B).

Example

For $m = 2$,

$$A = \begin{bmatrix} [x_0, x_1, x_2]a & [x_1, x_2]a & [x_2]a \\ [x_0, x_1]a & [x_1]a & 0 \\ [x_0]a & 0 & 0 \end{bmatrix}.$$

We need to show that the three submatrices

$$A_0 = \left[[x_0, x_1, x_2]a \right], \qquad A_1 = \begin{bmatrix} [x_0, x_1, x_2]a & [x_1, x_2]a \\ [x_0, x_1]a & [x_1]a \end{bmatrix},$$

and $A_2 = A$ are non-singular.

Proof of solvability

Since

$$a(x) = (x - u_0)(x - u_1)\cdots(x - u_\mu),$$

and since the grids are interlaced, for example,

$$u_0 < x_0 < u_1 < x_1 < \cdots ,$$

it follows that the values $a(x_0), a(x_1), \ldots, a(x_m)$ alternate in sign.

Consider first A_0

A_0 consists of the single element

$$[x_0, \ldots, x_m]a = \sum_{k=0}^{m} q_k,$$

where

$$q_k = a(x_k) \prod_{\substack{j=0 \\ j \neq k}}^{m} (x_k - x_j)^{-1}.$$

Since $x_0, x_1, \ldots, x_m$ are increasing, the product alternates in sign with k, and it follows that $q_0, q_1, \ldots, q_m$ have the same sign, and so

$$[x_0, \ldots, x_m]a \neq 0,$$

and A_0 is non-singular.

General submatrix A_r

For general r, there is an explicit formula for the determinant of A_r.

Lemma

For $r = 0, 1, \ldots, m$,

$$\det A_r = \sum_{0 \le k_0 < k_2 < \cdots < k_r \le m} D^2_{k_0,k_1,\ldots,k_r}\, q_{k_0} q_{k_1} \cdots q_{k_r},$$

and

$$D_{k_0,k_1,\ldots,k_r} = \prod_{0 \le i < j \le r} (x_{k_j} - x_{k_i}).$$

Since $q_0, \ldots, q_m$ have the same sign, $\det A_r \neq 0$.

Example

When $m = 2$,

$$q_0 = \frac{a(x_0)}{(x_0 - x_1)(x_0 - x_2)}, \quad q_1 = \frac{a(x_1)}{(x_1 - x_0)(x_1 - x_2)}, \quad q_2 = \frac{a(x_2)}{(x_2 - x_0)(x_2 - x_1)}.$$

The determinant of A_1 is

$$\begin{aligned}
&\begin{vmatrix} [x_0, x_1, x_2]a & [x_1, x_2]a \\ [x_0, x_1]a & [x_1]a \end{vmatrix} \\
&\quad = \begin{vmatrix} q_0 + q_1 + q_2 & (x_1 - x_0)q_1 + (x_2 - x_0)q_2 \\ (x_0 - x_2)q_0 + (x_1 - x_2)q_1 & (x_1 - x_0)(x_1 - x_2)q_1 \end{vmatrix} \\
&\quad\quad = (x_1 - x_0)^2 q_0 q_1 + (x_2 - x_0)^2 q_0 q_2 + (x_2 - x_1)^2 q_1 q_2.
\end{aligned}$$

Proof in general case

$A_r = [a_{j,m-i}]_{i,j=0,\dots,r}$, and we can express the elements as

$$a_{j,m-i} = \sum_{k=0}^{m} \hat{\phi}_i(x_k)\phi_j(x_k)q_k,$$

where

$$\phi_j(x) = (x - x_0)(x - x_1)\cdots(x - x_{j-1}),$$
$$\hat{\phi}_i(x) = (x - x_m)(x - x_{m-1})\cdots(x - x_{m-i+1}).$$

Thus, $A_r = BC$ where

$$B = [\hat{\phi}_i(x_k)]_{i=0,\dots,r,k=0,\dots,m}, \quad C = [\phi_j(x_k)q_k]_{k=0,\dots,m,j=0,\dots,r}.$$

By the Cauchy-Binet theorem,

$$|A_r| = \sum_{0\le k_0<k_1<\cdots<k_r\le m} |\hat{\phi}_i(x_{k_j})|_{i,j=0,\dots,r}|\phi_j(x_{k_i})q_{k_i}|_{i,j=0,\dots,r},$$

and the proof is completed by observing that

$$|\hat{\phi}_i(x_{k_j})|_{i,j=0,\dots,r} = |\phi_j(x_{k_i})|_{i,j=0,\dots,r} = |x_{k_j}^i|_{i,j=0,\dots,r} = D_{k_0,k_1,\dots,k_r}.$$