Roulets
Rotational Anisotropic Wavelet Transform

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Outline

Motivation

Problem

- project with Micro-Epsilon GmbH & Co. KG
- in metal processing different cold rolls are used for producing metal bands with different thickness

Motivation

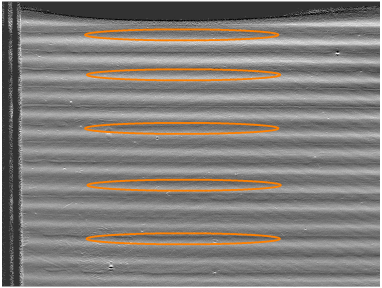

Figure: product surface with chatter marks

Problem

- project with Micro-Epsilon GmbH & Co. KG
- in metal processing different cold rolls are used for producing metal bands with different thickness
- chatter marks occur when cold rolls are defect
- detect defect cold roll out of characteristics

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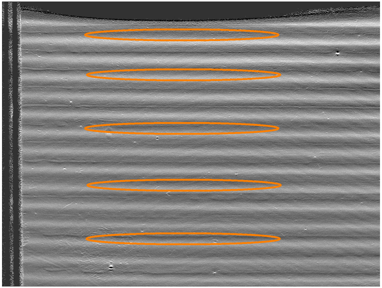


Figure: product surface with chatter marks

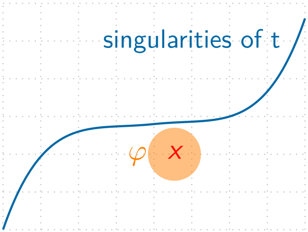
Aim

Detection of

- line singularities in pictures
- with integral transform

In the following:

- tempered distribution $t \in S'(\mathbb{R}^2)$
- x is a regular point
- $\varphi \in C_C^\infty(\mathbb{R}^2)$ cutoff function, identically one in neighbourhood of x



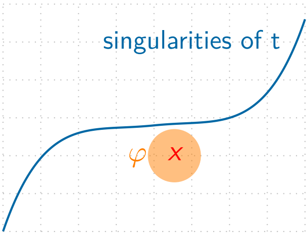
Line Singularities and Wavefront Set

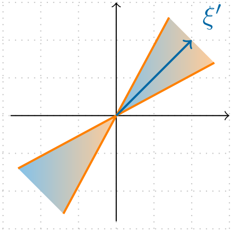
tempered distribution $t \in S'(\mathbb{R}^n)$
x is a regular point
φ cutoff function, identically one in an neighborhood of x

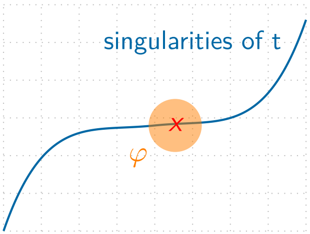
Definition

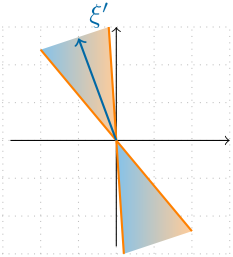
Then a pair $(x, \xi) \in \mathbb{R}^2 \times \mathbb{S}^1$ is a regular directed point if there exists cutoff function φ and a neighbourhood $W \subset \mathbb{S}^1$ of ξ such that **for all** $N \in \mathbb{N}$ there exists C_N with

$$\forall \xi' \in C(W) : |\widehat{\varphi t}(\xi')| \leq C_N (1 + |\xi'|)^{-N}.$$









Definition

The wavefront set $WF(t)$ is the complement of the regular directed points.

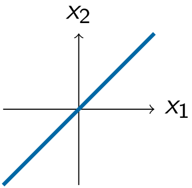


Figure: time domain

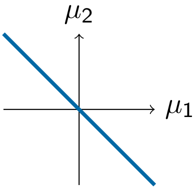


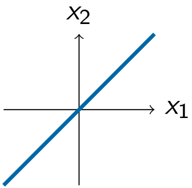
Figure: frequency domain

Wavefront set of line singularity

For $\delta_{x_2=p+qx_1}$ the wavefront set is

$$WF\left(\delta_{x_2=p+qx_1}\right) = \left\{(x_1, x_2)\,|x_2 = p + qx_1\right\} \times \left\{-\frac{1}{q}\right\}.$$

Line Singularities and Wavefront Set



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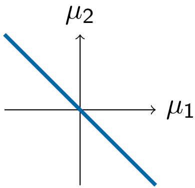

Figure: frequency domain

Aim

Integral transform that detects wavefront sets by coefficient decay rate.

Setting

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construction of wavelet like transform

Setting

construction of wavelet like transform

- quasi-regular representation

$$(\pi_L(a,b)f)(x) = \delta(a)^{-\frac{1}{2}} f\left(\sigma_{a^{-1}}(x \circ b^{-1})\right)$$

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$$\left(\pi_L(a,b)\,f\right)(x) = \delta(a)^{-\frac{1}{2}}\, f\left(\sigma_{a^{-1}}(x \circ b^{-1})\right)$$

- induces integral transform

$$\left(T_\varphi f\right)(a,b) = \langle f, \pi(a,b)\,\varphi\rangle$$

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When do integral transforms detect wavefront sets?

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When do integral transforms detect regular directed points?

Theory of Fell, Führ, Voigtlaender[1]

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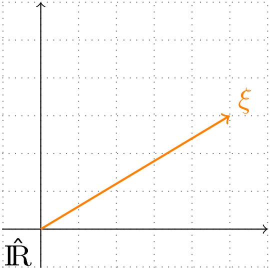
ξ direction



[1] Fell, Führ, Voigtlaender, Resolution of the wavefront set using general continuous wavelet transforms, J. Fourier Anal. Appl., 2014

Theory of Fell, Führ, Voigtlaender[1]

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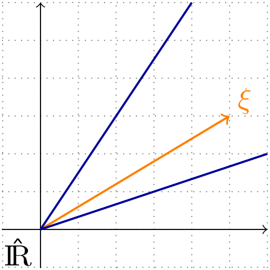

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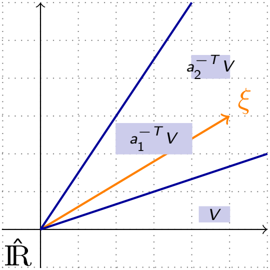

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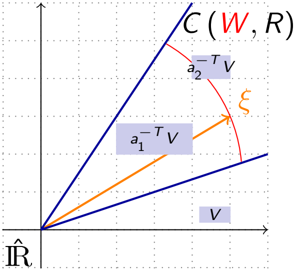
$R > 0$



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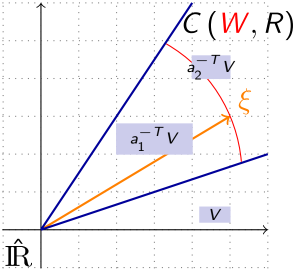

- transform has to distinguish between different ξ
- description with K_i and K_o

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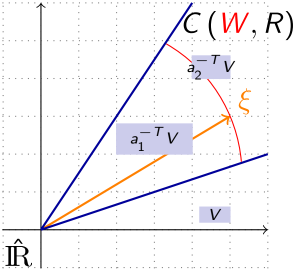

- transform has to distinguish between different ξ
- description with K_i and K_o
  - $K_i(V, W, R) := \{a \in A | a^{-T} V \subset C(W; R)\}$
  - $K_o(V, W, R) := \{a \in A | a^{-T} V \cap C(W; R) \neq \varnothing\}$

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Theory of Fell, Führ, Voigtlaender[2]

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Result

under appropriate conditions the following are equivalent

- (x, ξ) is a regular directed point of t
- local fast decay for $a \in K_o(W, V, R) : |W_{\psi_n} u(y, a)| \leq C_N ||a||^N$

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- characterisation with one wavelet
    1. shearlet group

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- characterisation with one wavelet
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- necessary to change wavelet for characterisation
  1. similitude group
  2. diagonal group

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} mixed case

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Basic Idea Behind Approach

construction of wavelet like transform

- quasi-regular representation

$$(\pi_L(a,b)f)(x) = \delta(a)^{-\frac{1}{2}} f\left(\sigma_{a^{-1}}(x \circ b^{-1})\right)$$

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  - $B_{tra} = \left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} : b_1, b_2 \in \mathbb{R} \right\}$

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- choices for matrix group A:
  - $A_{dil} = \left\{ \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} : s_1, s_2 \in \mathbb{R} \setminus \{0\} \right\}$
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  } composition: no group

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  } needle-like structure?
- choice for b:
  - $B_{tra} = \left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} : b_1, b_2 \in \mathbb{R} \right\}$

Heuristic: Action of A_{dil} in $\widehat{\mathbb{R}}^2$

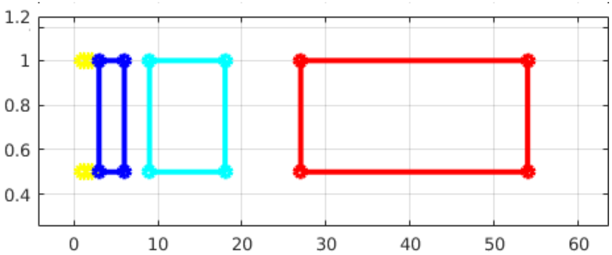
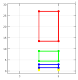
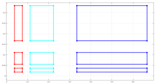


Figure: Scaling of $\begin{pmatrix} s_1 & 0 \\ 0 & 1 \end{pmatrix}$





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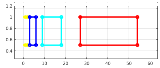


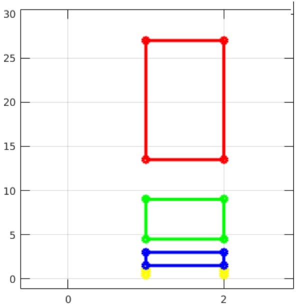
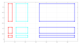


Figure: Scaling of $\begin{pmatrix} 1 & 0 \\ 0 & s_2 \end{pmatrix}$



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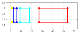

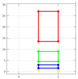

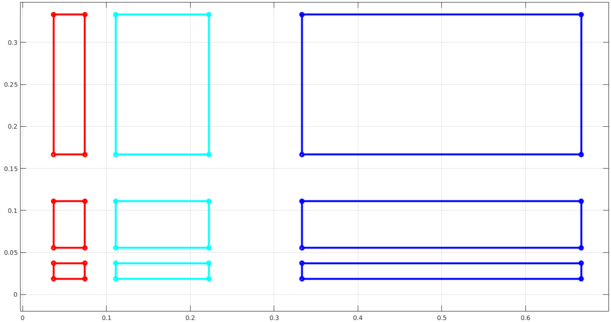

Figure: Scaling of $\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$

Heuristic: Action of $A_{dil}A_{rot}$ in $\widehat{\mathbb{R}}^2$

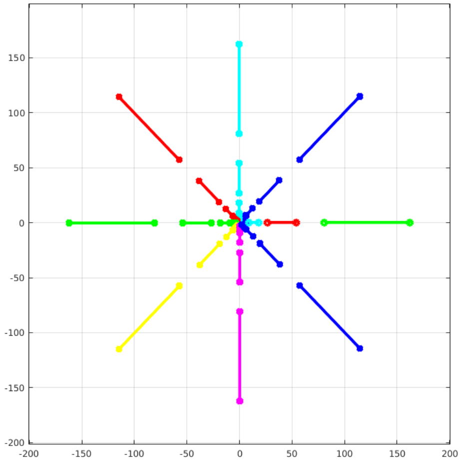

Figure: Three scalings in x and y direction with different rotations

Heuristic: Action of $A_{rot} A_{dil}$ in $\widehat{\mathbb{R}}^2$

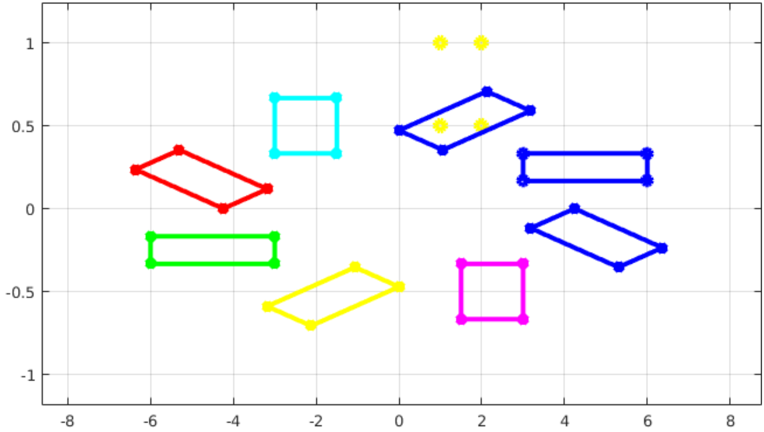

Figure: different rotations

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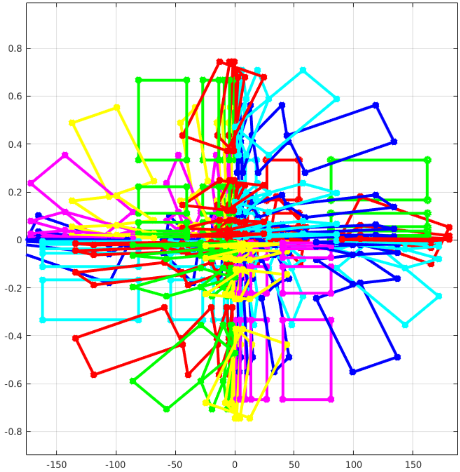

Figure: Different scalings in x and y direction with different rotations

Extended Definition

Sets K_i^T and K_o^T

$$K_i^T(W,V,R) := \left\{tm \middle| t \in T, m \in M, (tm)^{-T} V \subset C(W,R)\right\}$$
$$K_o^T(W,V,R) := \left\{tm \middle| t \in T, m \in M, (tm)^{-T} V \cap C(W,R) \neq \emptyset\right\}$$

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Proof possible for

- $A_{rot} A_{dil}$: fits not into setting
- $A_{dil} A_{rot}$: fits into setting

Roulet Transform

Definition

Consider

- $\psi \in L_2\left(\mathbb{R}^2\right)$ admissible for A_{sim}
- $f \in L_2\left(\mathbb{R}^2\right)$
- $s_1, s_2 \in \mathbb{R} \setminus \{0\}, \alpha \in [0, 2\pi)$ and $b \in \mathbb{R}^2$

The roulet transform is given by

$$R_\psi f\left(s, \alpha, b\right) = \int_{\mathbb{R}^2} f\left(x\right) \overline{\psi_{s,\alpha,b}\left(x\right)} dx,$$

with

$$\psi_{s,\alpha,b}\left(x\right) = \left|s_1 s_2\right|^{-\frac{1}{2}} \psi\left(R_\alpha^{-1} A_s^{-1}\left(x - b\right)\right).$$

Tensor product wavelets

$$\psi_1, \psi_2 \text{ are 1D wavelets} \Rightarrow \int_{\hat{\mathbb{R}}^2} \frac{|\hat{\psi}(\xi_1, \xi_2)|^2}{|\xi_1^2 + \xi_2^2|} d\xi_1 d\xi_2 < \infty,$$

where $\psi(x, y) = \psi_1(x)\psi_2(y)$

Frequency Domain

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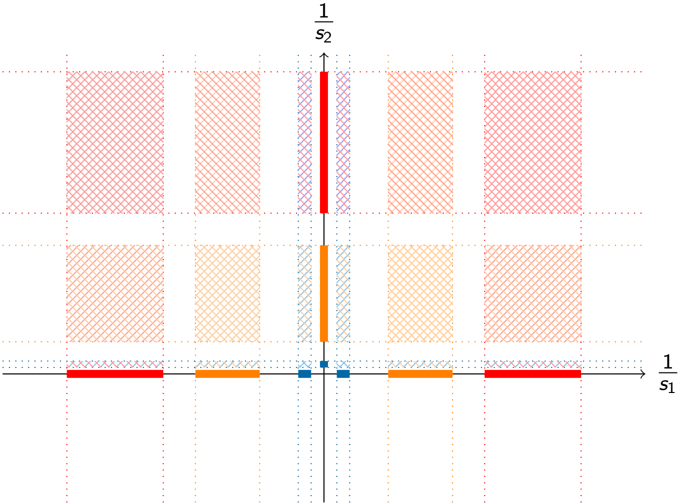

Figure: support of roulets

Line Singularities

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Theorem [G.]

Let $g(x) = \delta_{x_2 = qx_1}(x)$ for $q \neq 0$.
For $-b_1 = qb_2$, $p = \frac{\cos(\alpha)}{\sin(\alpha)}$ and $s_1 \geq \frac{s_2 \cos(\alpha)}{\cos(\alpha) - \sqrt{s_2}}$

$$RAW_\psi g(s, \alpha, b) \sim |s_1|^{-\frac{1}{2}}, \quad \text{for } s_1 \to 0,$$

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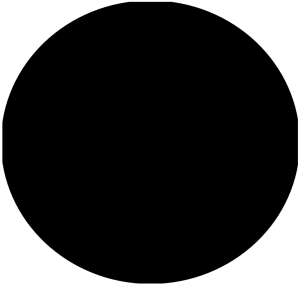
Remarks

1. depending on α: choose s_2 sufficient small
2. $RAW_{\psi} g(s, \alpha, b)$ is divergent on the singularity

Example - Circle



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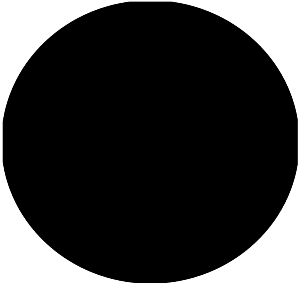

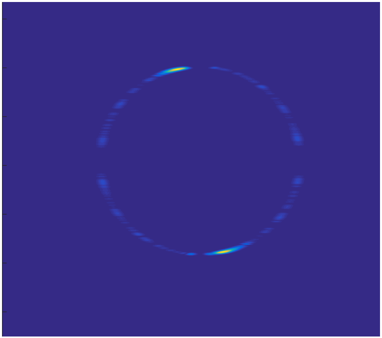

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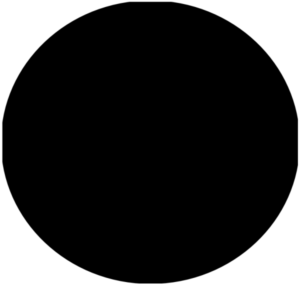

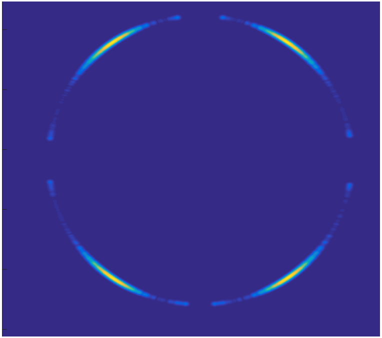

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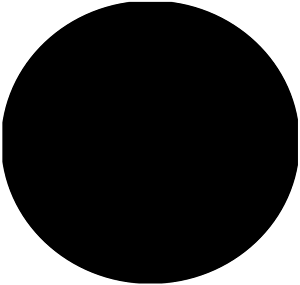

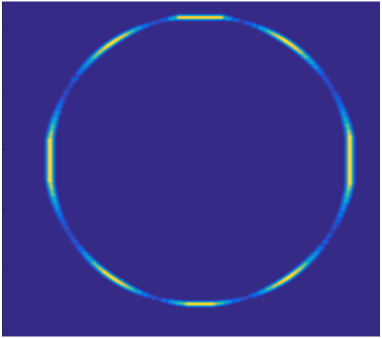

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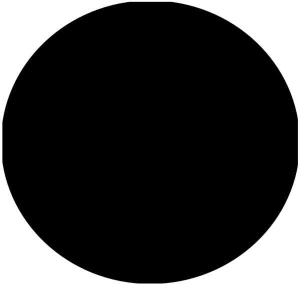

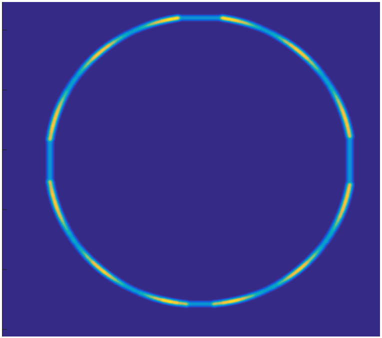

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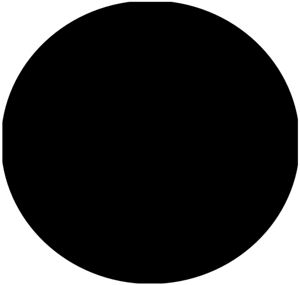
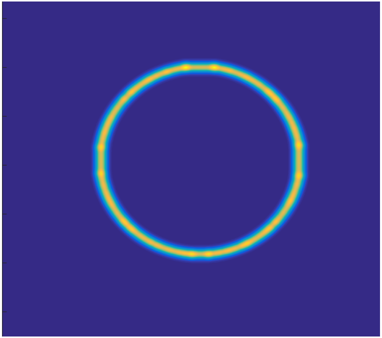
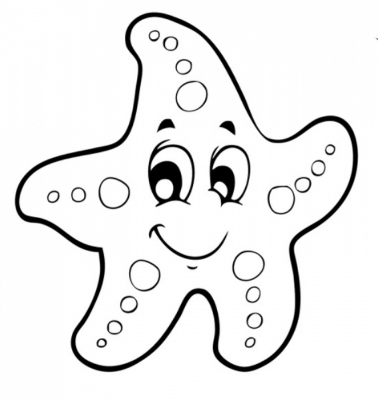




Example - Sea Star



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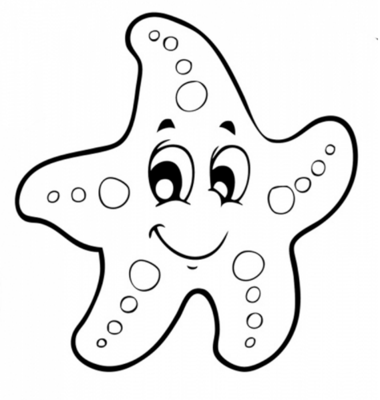

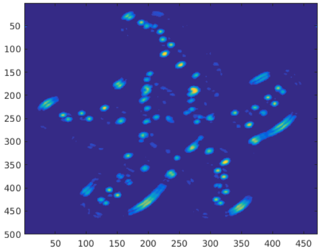

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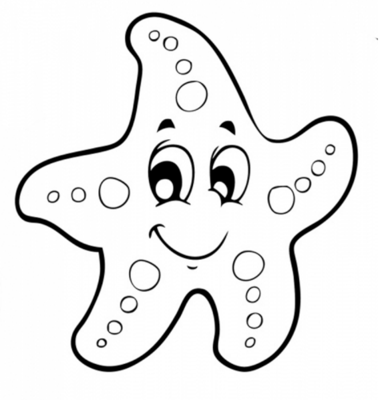

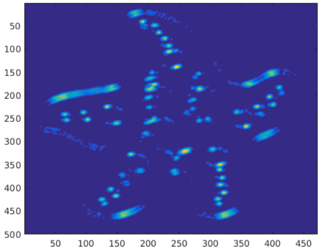

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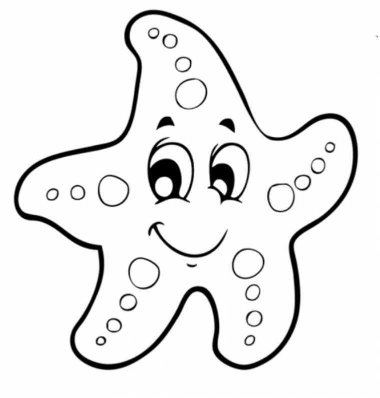

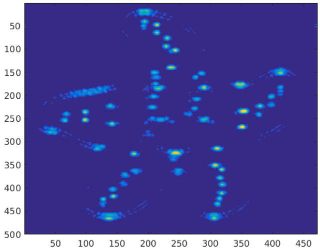

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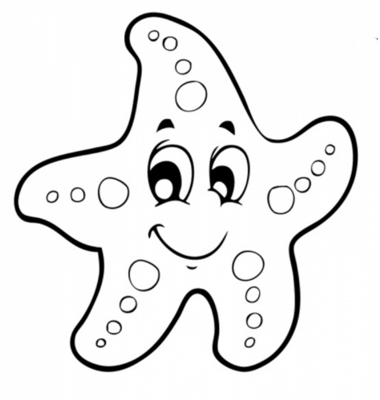

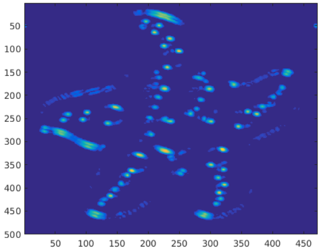

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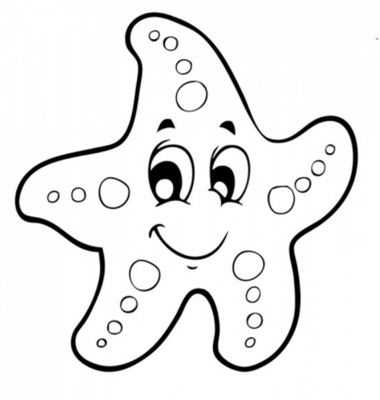

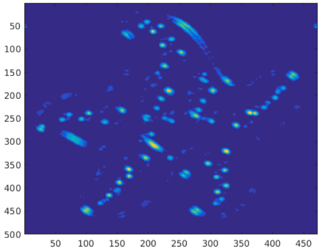

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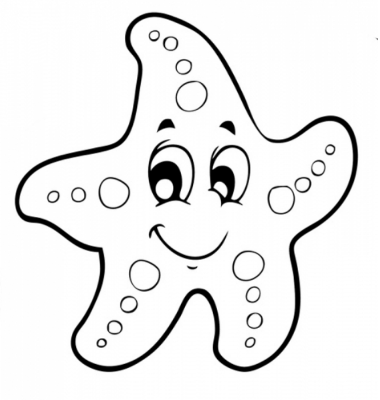

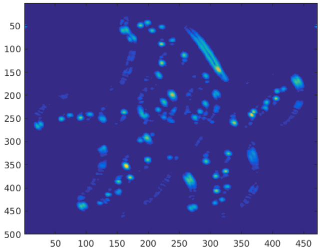

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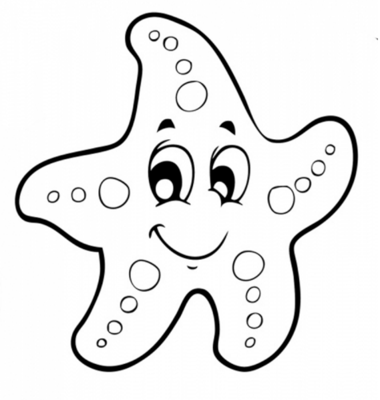

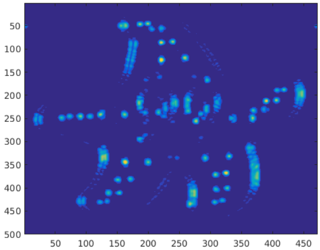

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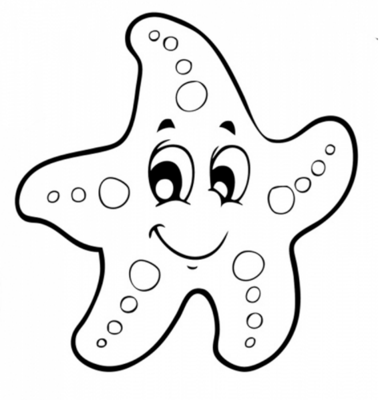

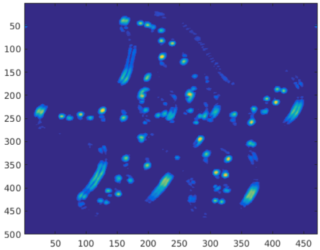

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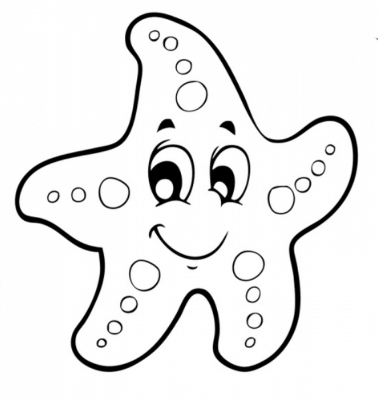
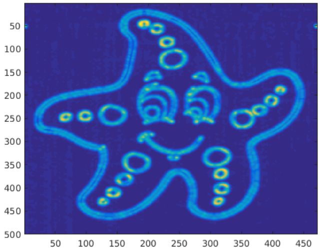
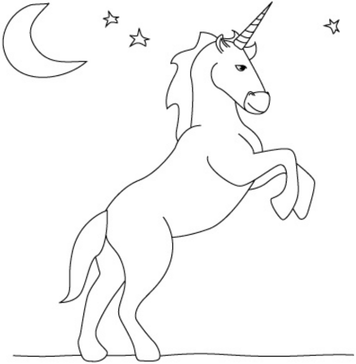




Example - Unicorn



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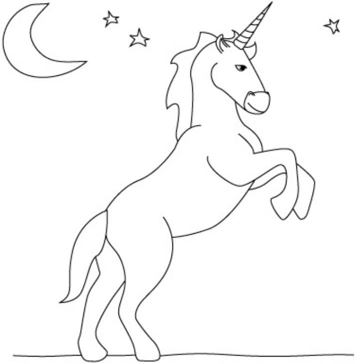

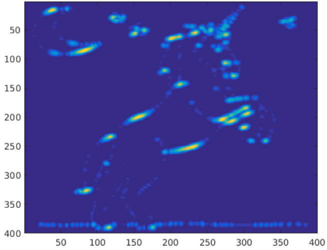

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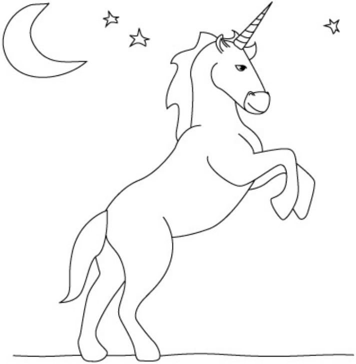

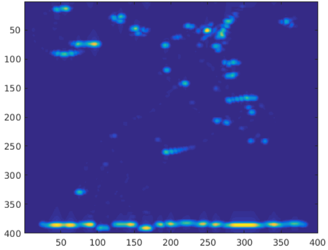

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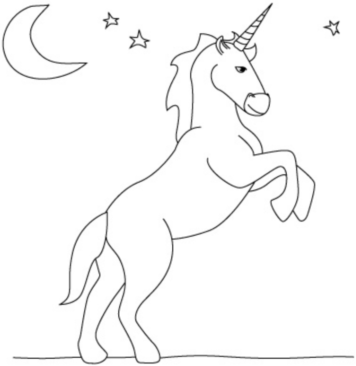

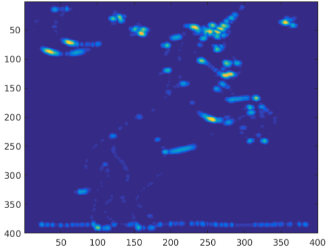

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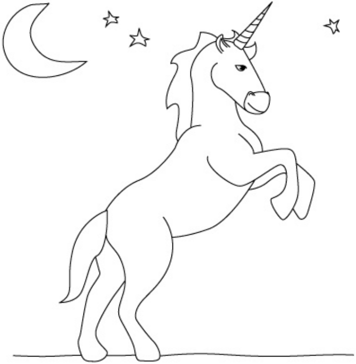

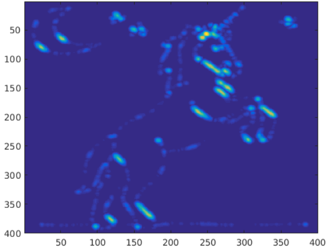

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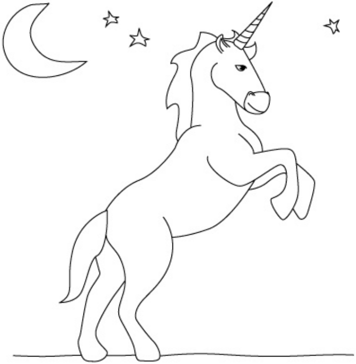

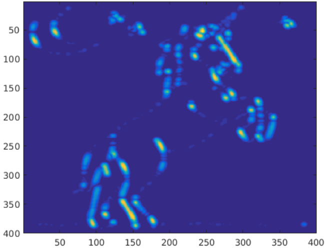

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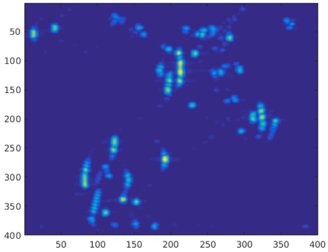

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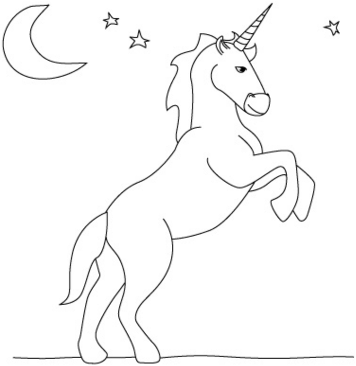

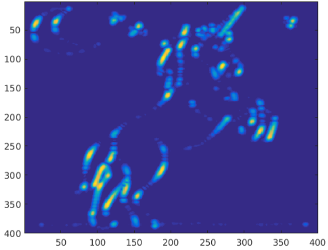

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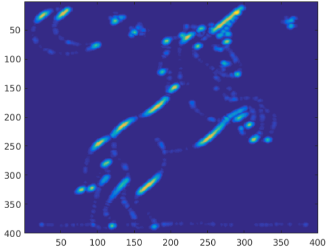

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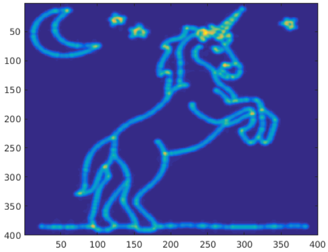
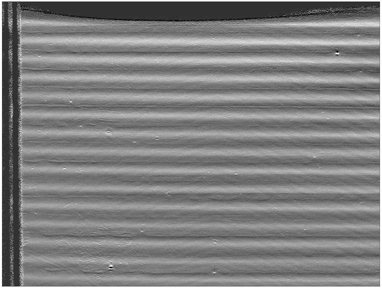
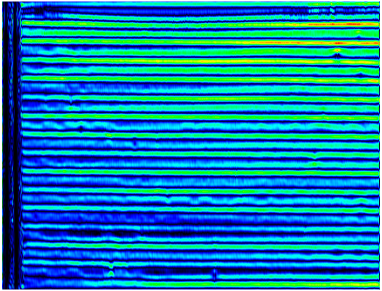


Back to Application





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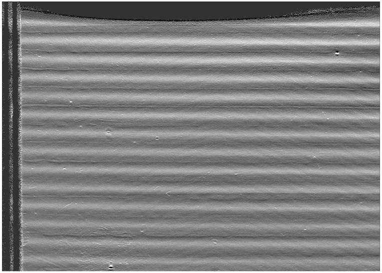

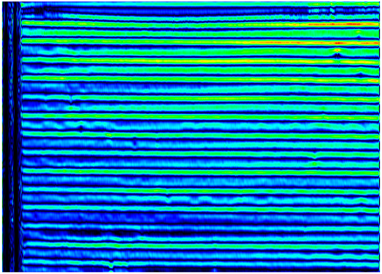

Improvements

- wavelet like transform with rotation and anisotropic scaling
- detection of line singularities
- fast implementation with FFT

UNIVERSITÄT PASSAU
Fakultät für Informatik und Mathematik

Thank you for your attention!