Bernried, Germany, February 25- March 3, 2017
2nd IM-Workshop on "Applied Approximation, Signals and Images"

A class of nonstationary biorthogonal wavelet filters

V. Bruni[1], M. Cotronei [2], F. Pitolli[1]

[1]Università di Roma "La Sapienza"

[2] Università Mediterranea di Reggio Calabria

Outline

- **Construction** *of a class of biorthogonal wavelet filters associated to a Nonstationary (NS) Multiresolution Analysis (MRA)*

  ⇒ *scale-dependent filters*

- **Test** *of nonstationary wavelet filters in image processing applications*

- **Guidelines** *for future research*

Nice wavelet properties

- Compact support ⟹ local information
- Vanishing moments ⟹ detection of singularities
- MRA (filter bank) ⟹ fast implementation

⇓

New nonstationary wavelet families

change of support length and/or number of vanishing moments at any scale level while preserving MRA

[Conti-Gori-P 2007, Gori-P 2008, P 2016]

Nonstationary refinement masks

The mask coefficients $\{a_\alpha^{(n,j)}, \alpha \in Z\}$ depend on the scale level j through the real parameter μ

$$\begin{cases} a_\alpha^{(n,0)} = \dfrac{1}{2^{n-2}} \dbinom{n-1}{\alpha-1}, & j=0 \\ a_\alpha^{(n,j)} = \dfrac{1}{2^{n+\frac{1}{j^\mu}}} \left[\dbinom{n+1}{\alpha} + 4\left(2^{\frac{1}{j^\mu}}-1\right) \dbinom{n-1}{\alpha-1} \right], & j \ge 1 \end{cases}$$

For fixed μ and increasing j (scale level), the filter taps change:

they are concentrated in a small support for small j,

while they converge to the limit mask $\{\lim_{j\to\infty} a_\alpha^{(n,j)}\}$ for large j

but

the number of vanishing moments is fixed except for the limit mask

Nonstationary refinement masks

Example: $n = 3$

$$a^{(3,0)} = \frac{1}{2}\{0,1,2,1,0\}$$ (degree **1** B-spline mask)

$$a^{(3,j)} = \frac{1}{2^3}\left\{\frac{1}{2^{\frac{1}{j^\mu}}}, 4, 8 - \frac{2}{2^{\frac{1}{j^\mu}}}, 4, \frac{1}{2^{\frac{1}{j^\mu}}}\right\}$$

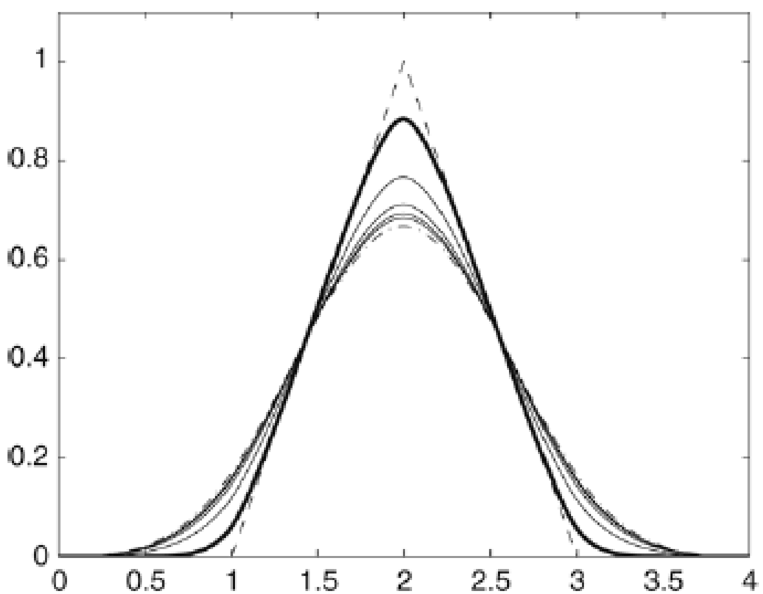
$\downarrow$ $j \to +\infty$

$$a^{(3,\infty)} = \frac{1}{2^3}\{1,4,6,4,1\}$$ (degree **3** B-spline mask)

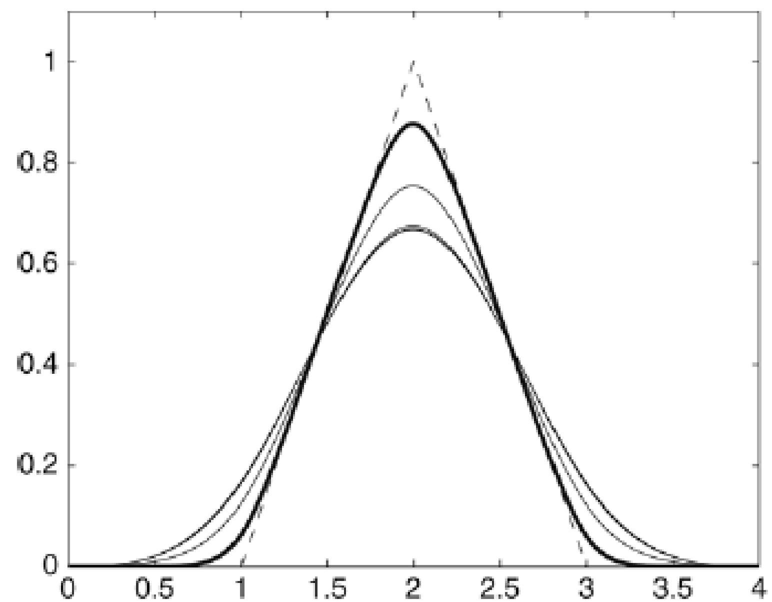
Nonstationary refinable functions

The higher μ the faster the convergence
to the smoother B-spline

$\mu = 1.5$



$\mu = 4$



Nonstationary refinement symbols

- The symbol of any mask is defined as the Laurent polynomial

$$a^{(n,j)}(z) = \sum_{\alpha \in \mathbf{Z}} a_{\alpha}^{(n,j)} \, z^{\alpha}$$

$$\Downarrow$$

$$a^{(n,0)}(z) = \frac{1}{2^{n-2}} (1+z)^{n-1} z$$

$$a^{(n,j)}(z) = \frac{1}{2^{n+\frac{1}{j^{\mu}}}} (1+z)^{n-1} \left[z^2 + (2^{\frac{1}{j^{\mu}}+2} - 2) z + 1 \right] \quad j \geq 1$$

Biorthogonal nonstationary filters

$$f = \sum_k c_k^{(m)} \varphi(2^m \cdot - k)$$

$$f = Q_{m-1}f + Q_{m-2}f + \ldots + Q_{m-L}f + P_{m-L}f$$

where:

$$P_j f = \sum_k c_k^{(j)} \varphi_k^{(j)}, \quad Q_j f = \sum_k d_k^{(j)} \psi_k^{(j)}$$

with $\quad c_k^{(j)} = \langle f, \widetilde{\varphi}_k^{(j)} \rangle, \quad d_k^{(j)} = \langle f, \widetilde{\psi}_k^{(j)} \rangle$

since $\quad \widetilde{\varphi}_{,h}^{(j)} = \sum_k \widetilde{a}_{k-2h}^{(j)} \widetilde{\varphi}_k^{(j+1)}, \quad \widetilde{\psi}_{,h}^{(j)} = \sum_k \widetilde{b}_{k-2h}^{(j)} \widetilde{\varphi}_k^{(j+1)}$

⇓

Non stationary

decomposition scheme

$$c_{,h}^{(j)} = \sum_k \tilde{a}_{k-2h}^{(j)} c_k^{(j+1)}$$

$$d_{,h}^{(j)} = \sum_k \tilde{b}_{k-2h}^{(j)} c_k^{(j+1)}$$

reconstruction scheme

$$c_k^{(j+1)} = \sum_h (a_{k-2h}^{(j)} c_{,h}^{(j)} + b_{k-2h}^{(j)} d_{,h}^{(j)})$$

Biorthogonal nonstationary filters

- Set N = n-1 and $\beta = 2^{j^{-\mu}}$

Express the symbol in terms of trigonometric polynomials

$$a_N(\omega;\beta) = e^{-i\varepsilon\omega/2}\cos\left(\frac{\omega}{2}\right)^N L(\cos\omega;\beta)$$

where $\varepsilon = 0$ for N even, $\varepsilon = 1$ for N odd and

$$L(\cos\omega;\beta) = \frac{\cos\omega + 2\beta - 1}{\beta}$$

Express the dual in the same form

$$\widetilde{a}_N^{\widetilde{N}}(\omega;\beta) = e^{-i\varepsilon\omega/2}\cos\left(\frac{\omega}{2}\right)^{\widetilde{N}} \widetilde{L}(\cos\omega;\beta),$$

?

Biorthogonal nonstationary filters

From the biorthogonality condition in terms of symbols

$$a(z;\beta)\tilde{a}(z^{-1};\beta)+a(-z;\beta)\tilde{a}(-z^{-1};\beta)=4, \qquad z\in C\setminus\{0\}$$

$\tilde{L}(\cos\omega;\beta)$ satisfies the Bézout identity

$$(1-y)^q\,\frac{\beta-y}{2\beta}\,\tilde{L}(y;\beta)+y^q\,\frac{\beta-1+y}{2\beta}\,\tilde{L}(1-y;\beta)=1$$

where $y=\sin^2(\omega/2)$ and $q=\dfrac{N+\tilde{N}}{2}$

Biorthogonal nonstationary filters

1. Rewrite the Bézout identity

$$\widetilde{L}(y;\beta) = (1-y)^{-q}\frac{2\beta}{\beta-y} - (1-y)^{-q}\frac{2\beta}{\beta-y}y^q\frac{\beta-1+y}{2\beta}\widetilde{L}(1-y;\beta)$$

2. Consider the Taylor expansion of the right hand side with respect to y to compute the first q terms of

$$\widetilde{L}(y;\beta) = \sum_{k=0}^{q}\widetilde{l}_k y^k,$$

where $\widetilde{l}_k = \widetilde{l}_k(\beta)$

They are the first q terms of

$$(1-y)^{-q}\frac{2\beta}{\beta-y} = 2\sum_{k=0}^{q-1}\left(\sum_{h=0}^{k}\binom{q+h-1}{h}\left(\frac{1}{\beta}\right)^{k-h}\right)y^k + y^q R(y)$$

Biorthogonal nonstationary filters

- hence

$$\widetilde{l}_k = \frac{2}{\beta^k} \sum_{h=0}^{k} \binom{q+h-1}{h} \beta^h, \quad k = 0, \ldots, q-1$$

3. Compute the (q+1)-th term: the term y^q of the Taylor expansion

$$\widetilde{l}_q = \frac{2}{\beta^q} \sum_{h=0}^{q} \binom{q+h-1}{h} \beta^h - \frac{\beta-1}{\beta} \sum_{k=0}^{q} \widetilde{l}_k$$

that can be rewritten as

$$\widetilde{l}_q = 2\binom{2q-1}{q} + \frac{1}{\beta}\widetilde{l}_{q-1} - \frac{\beta-1}{\beta} \sum_{k=0}^{q} \widetilde{l}_k$$

Biorthogonal nonstationary filters

that is

$$\widetilde{l}_q = \frac{2}{2\beta - 1}\widetilde{l}_{q-1}$$

and then the dual symbol is

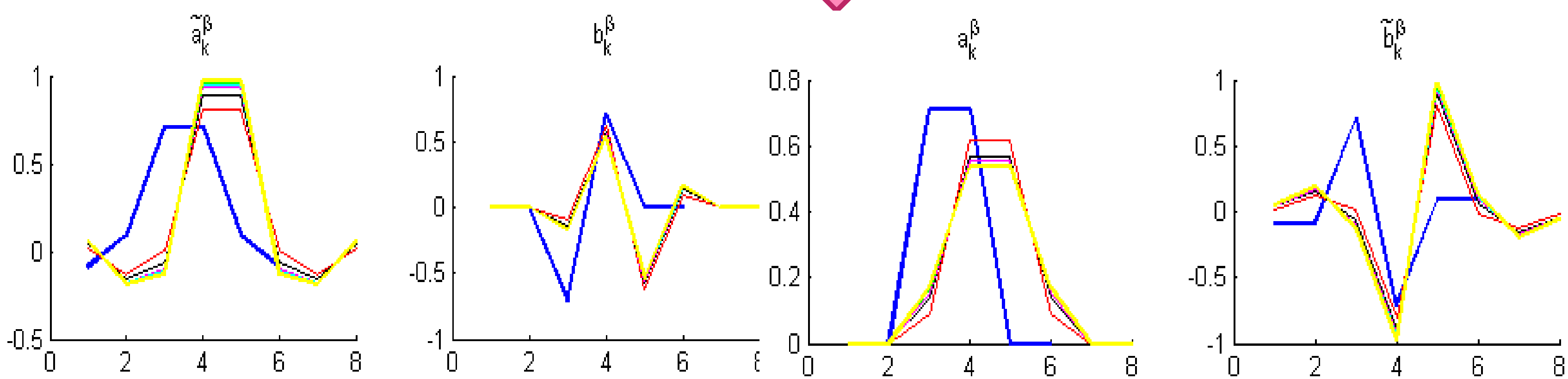
$$\widetilde{a}(z;\beta) = \frac{1}{z^{(N-\varepsilon)/2}}\left(\frac{1+z}{2}\right)^{\widetilde{N}} \sum_{k=0}^{q} \frac{\widetilde{l}_k}{2^{2k}}\left(2 - \frac{1}{z} - z\right)^k$$

Biorthogonal nonstationary filters

For example, for N=1

	k	$\tilde{N}=1$	$\tilde{N}=3$	$\tilde{N}=5$
$\{\tilde{a}_k^\beta\}$	$1,\ 0$	$-1+4\,\beta$	$-18\,\beta+64\,\beta^2-1$	$-22\,\beta+1024\,\beta^3-300\,\beta^2-2$
	$2,-1$	-1	$-1-14\,\beta+8\,\beta^2$	$176\,\beta^3-212\,\beta^2-14\,\beta-2$
	$3,-2$		$-(1+2\,\beta)\,(-1+4\,\beta)$	$-176\,\beta^3-38\,\beta^2+17\,\beta+3$
	$4,-3$		$1+2\,\beta$	$-24\,\beta^3+38\,\beta^2+21\,\beta+3$
	$5,-4$			$(-1+4\,\beta)\,(6\,\beta^2+3\,\beta+1)$
	$6,-5$			$-6\,\beta^2-3\,\beta-1$
	Denomin.	$2(2\beta-1)$	$2^5\beta(2\beta-1)$	$2^9\beta^2(2\beta-1)$

j=0,…,6 μ=1.5



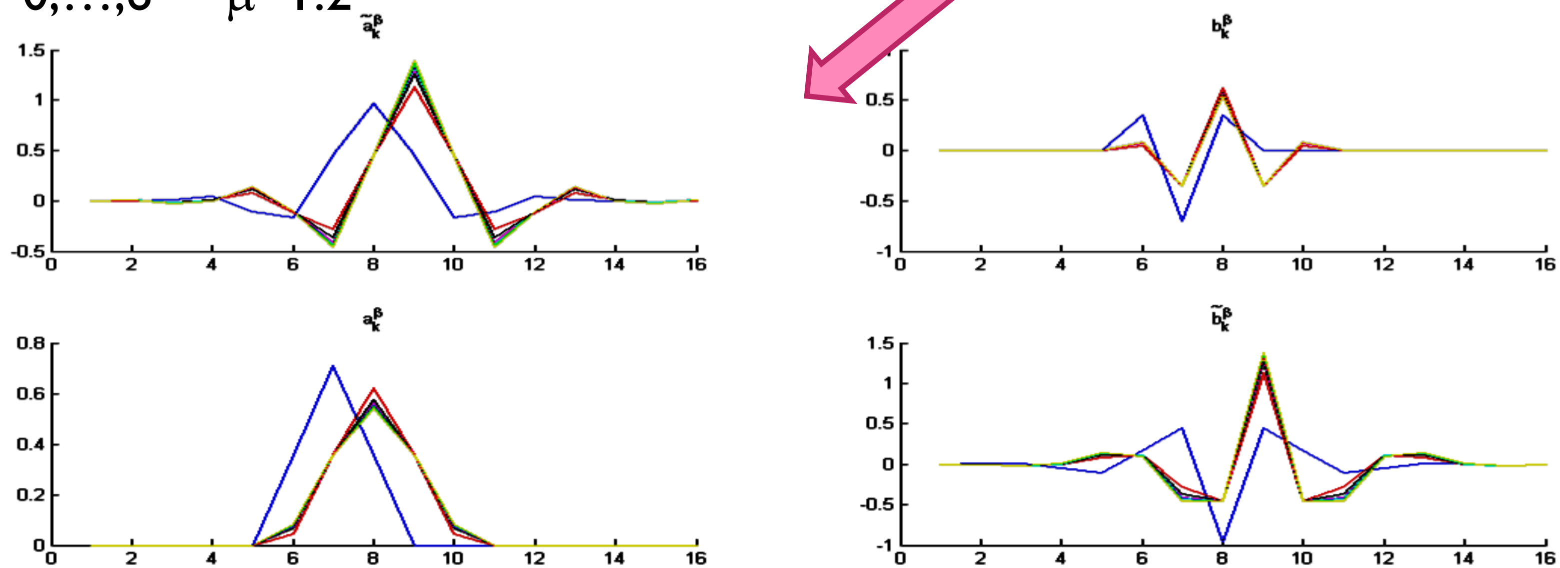
Biorthogonal nonstationary filters

For example, for N=2

$\{\tilde{a}_k^{\beta}\}$

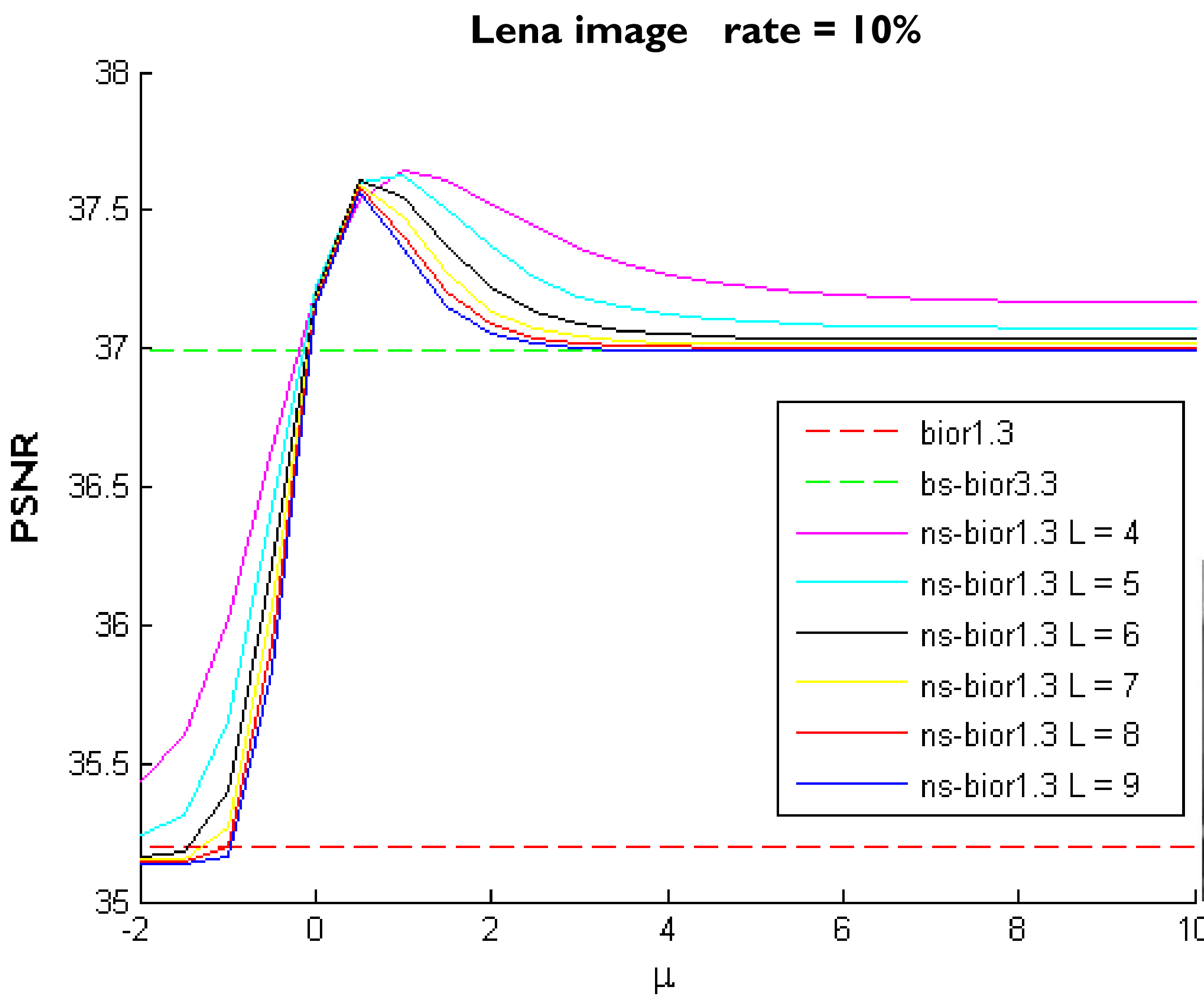
k	$\widetilde{N}=2$	$\widetilde{N}=4$	$\widetilde{N}=6$
0	$8\,(6\,\beta-1)\,\beta$	$8\,(90\,\beta^2-19\,\beta-1)\,\beta$	$16\,\beta\,(-1+700\,\beta^3-162\,\beta^2-12\,\beta)$
$1,-1$	$16\,\beta^2-10\,\beta-1$	$-14\,\beta-2-148\,\beta^2+304\,\beta^3$	$-5-2308\,\beta^3-210\,\beta^2+5184\,\beta^4-36\,\beta$
$2,-2$	$-4\,(2\,\beta+1)\,\beta$	$-64\,\beta^2\,(2\,\beta+1)$	$-4\,(492\,\beta^3+246\,\beta^2-4\,\beta-1)\,\beta$
$3,-3$	$2\,\beta+1$	$26\,\beta^2-48\,\beta^3+17\,\beta+3$	$9+60\,\beta-1248\,\beta^4+356\,\beta^3+266\,\beta^2$
$4,-4$		$4\,(6\,\beta^2+3\,\beta+1)\,\beta$	$8\,(68\,\beta^3+34\,\beta^2+12\,\beta+1)\,\beta$
$5,-5$		$-1-6\,\beta^2-3\,\beta$	$160\,\beta^4-116\,\beta^3-66\,\beta^2-28\,\beta-5$
$6,-6$			$-4\,\beta\,(10\,\beta^2+4\,\beta+1+20\,\beta^3)$
$7,-7$			$10\,\beta^2+4\,\beta+1+20\,\beta^3$
Denomin.	$2^4\beta(2\beta-1)$	$2^8\beta^2(2\beta-1)$	$2^{12}\beta^3(2\beta-1)$

j=0,…,6 μ=1.2



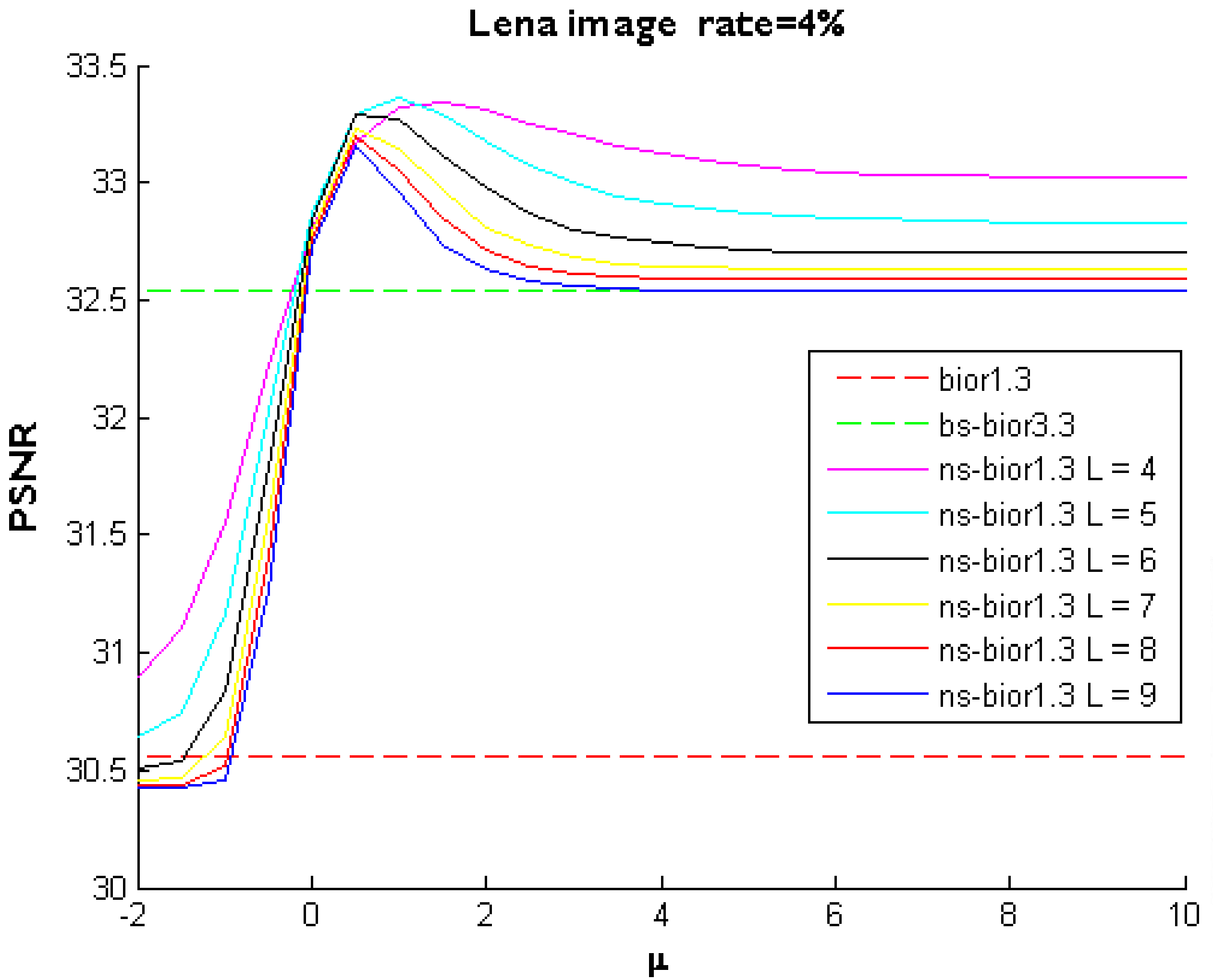
Some results

Compaction Properties: better PSNR at the same rate ($PSNR = 10 \log_{10} \frac{255^2}{\sqrt{MSE}}$)



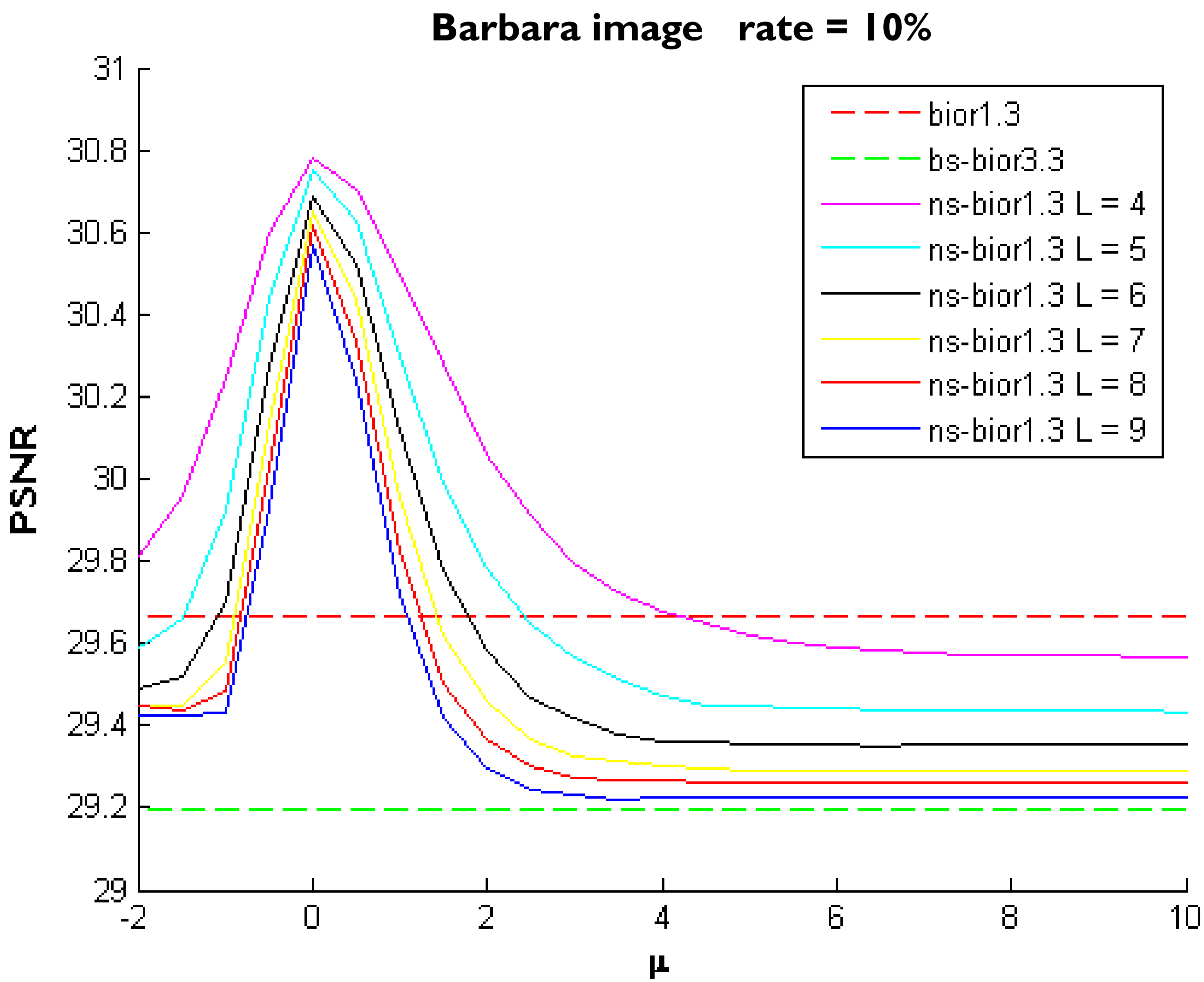
Some results

- Compaction Properties: better PSNR at the same rate



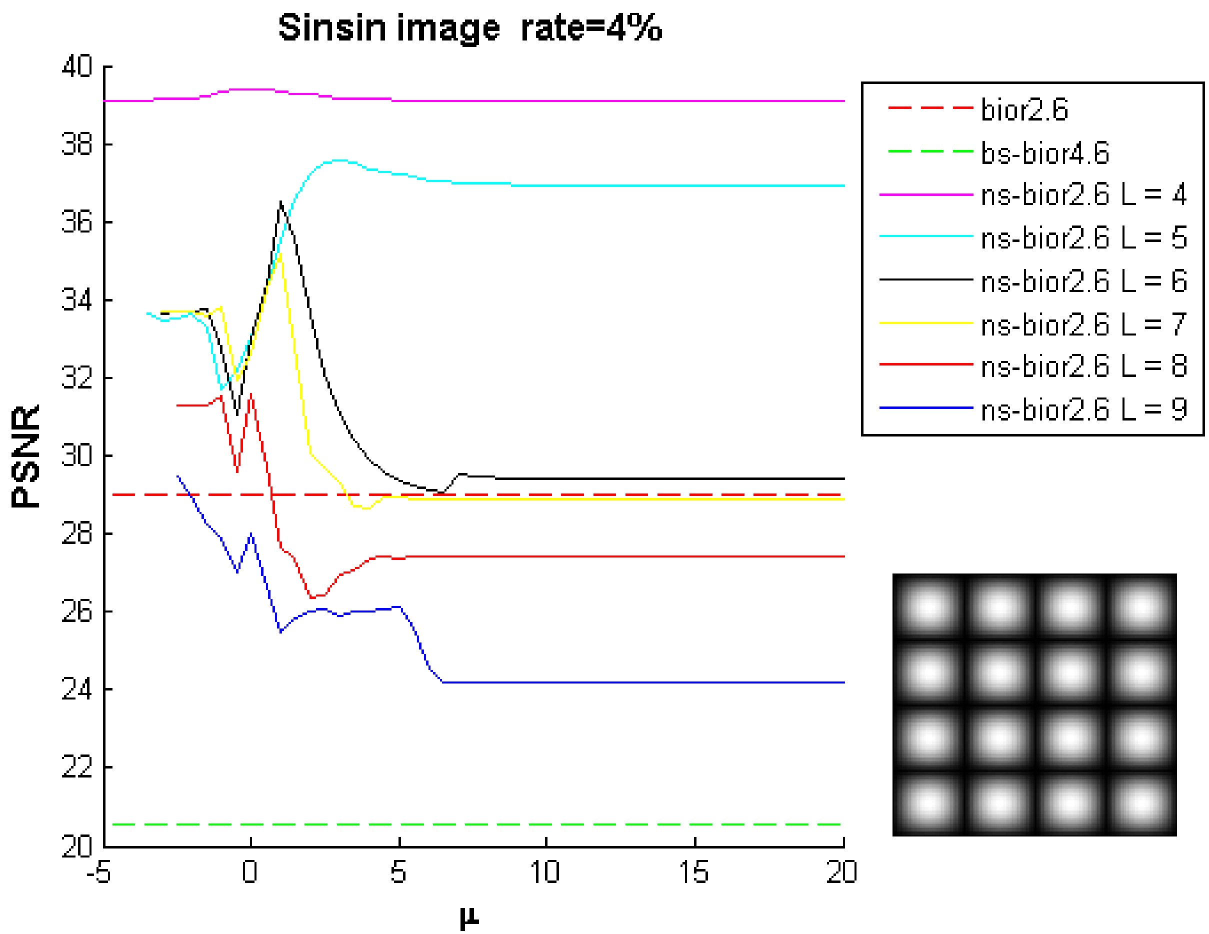
Some results

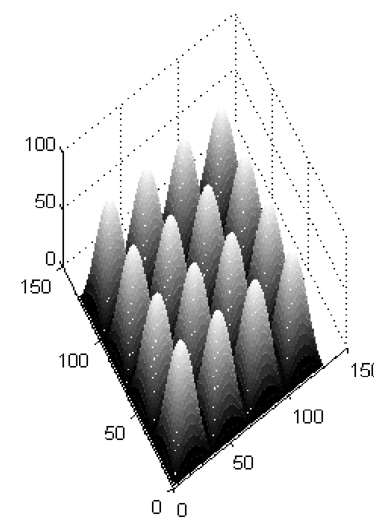
- Compaction Properties: better PSNR at the same rate



Some results

- Compaction Properties: better PSNR at the same rate





Some applications: image compression

Comparison with SPIHT : **comparable PSNR**

rate = 0.50 bpp

PSNR = 30.29 db

ns-bior1.3

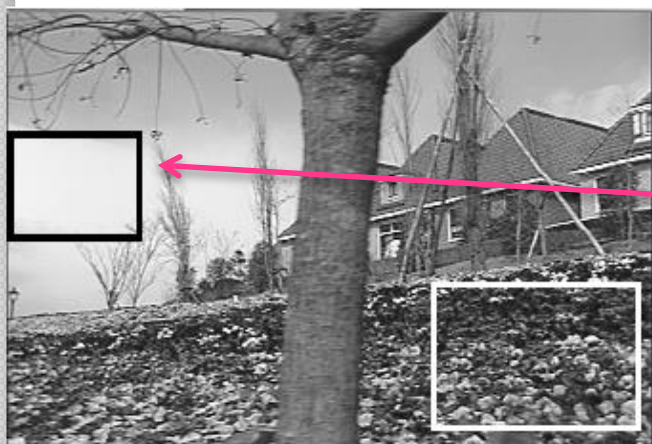
PSNR = 30.50 db

SPIHT

Some applications: image compression

The Structural SIMilarity index

$$SSIM(I,J) = \underbrace{\frac{2\mu_I\mu_J + C_1}{\mu_I^2 + \mu_J^2 + C_1}}_{luminance\ adaptation} \; \underbrace{\frac{2\sigma_I\sigma_J + C_2}{\sigma_I^2 + \sigma_J^2 + C_2}}_{contrast\ masking} \; \underbrace{\frac{\sigma_{IJ} + C_3}{\sigma_I\sigma_J + C_3}}_{spatial\ correlation},$$

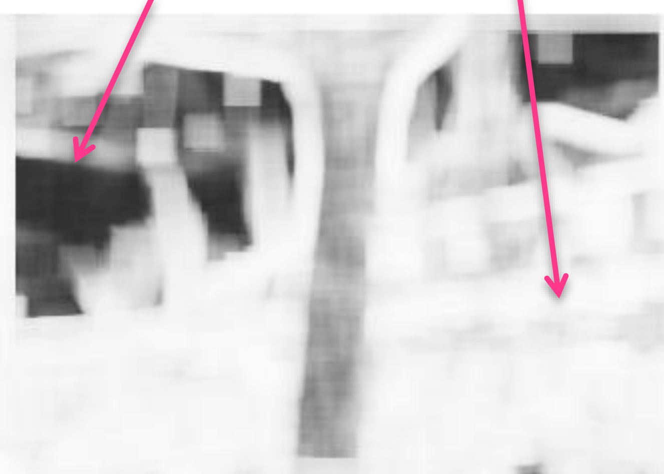


well correlated with Human Perception

same PSNR

SSIM = 0.3941

SSIM = 0.9570



Some applications: image compression

Comparison with SPIHT : **comparable PSNR and better visual quality**

PSNR = 30.29 db
SSIM = 0.8808
ns-bior1.3

rate = 0.50 bpp

PSNR = 30.50 db
SSIM = 0.8689
SPIHT

Some applications: image compression

Comparison with SPIHT : **comparable PSNR and better visual quality**

PSNR = 27.53 db
SSIM = 0. 8148
ns-bior1.5

rate = 0.30 bpp
L=6

PSNR = 27.71 db
SSIM = 0. 8044
SPIHT

Some applications: image compression

Comparison with SPIHT : **smaller PSNR and comparable visual quality**

rate = 0.20 bpp

ns-bior1.5

PSNR = 31.59 db
SSIM = 0. 8690

SPIHT

PSNR = 32.25 db
SSIM = 0. 8693

Some applications: image compression

Comparison with SPIHT : **smaller PSNR and comparable visual quality**

PSNR = 31.49 db
SSIM = 0. 8689

rate = 0.20 bpp

PSNR = 32.25 db
SSIM = 0. 8693

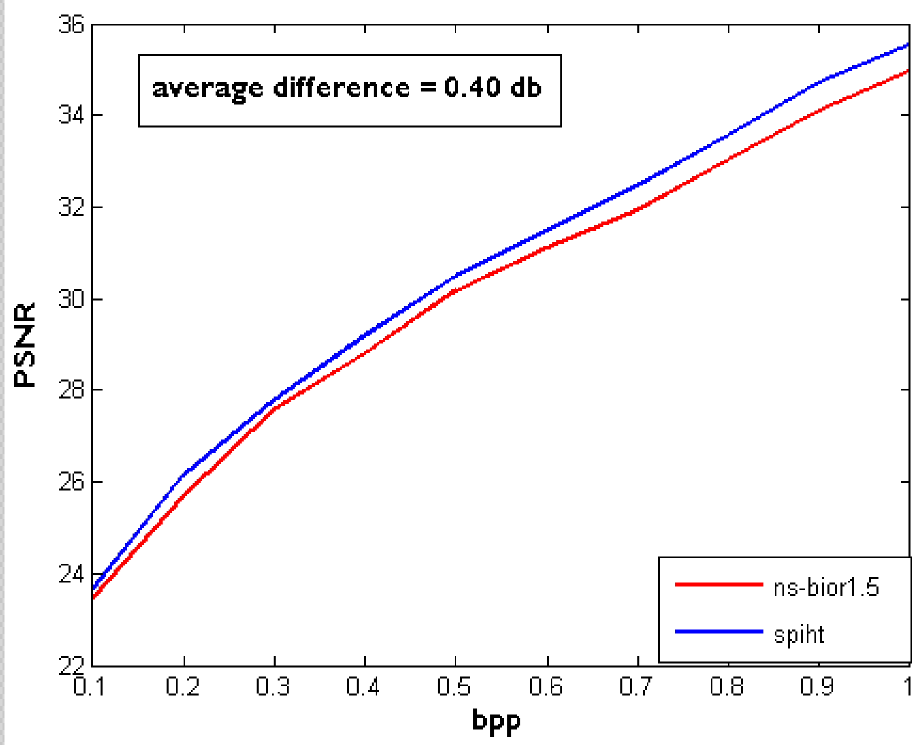
ns-bior2.4

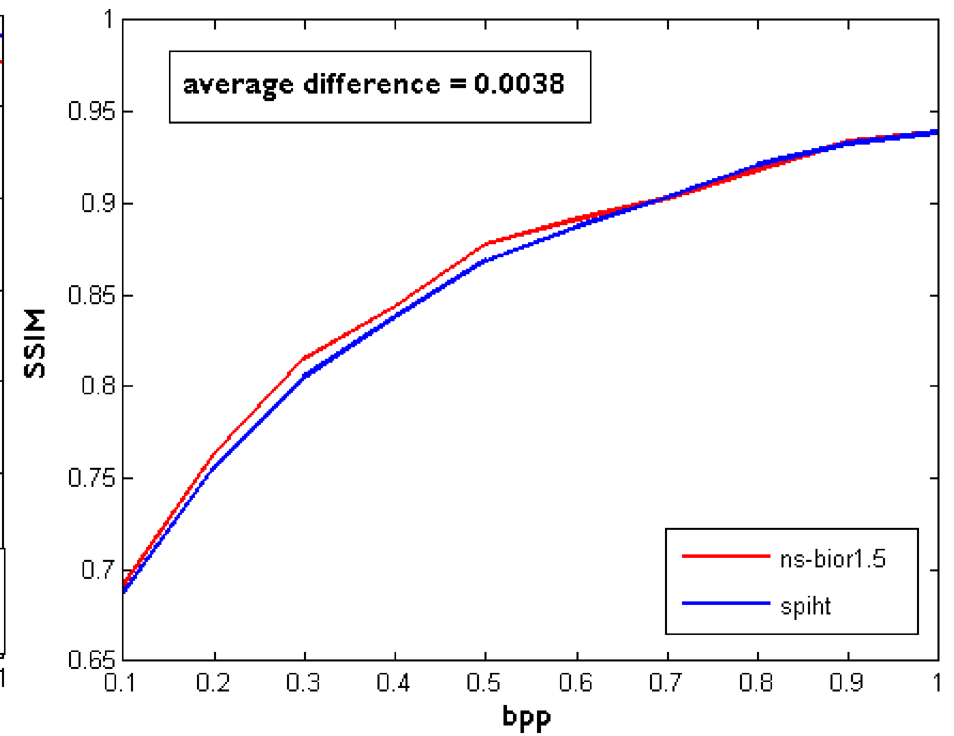
SPIHT

Some applications: image compression

Comparison with SPIHT : **comparable PSNR and better visual quality**

Cameraman image

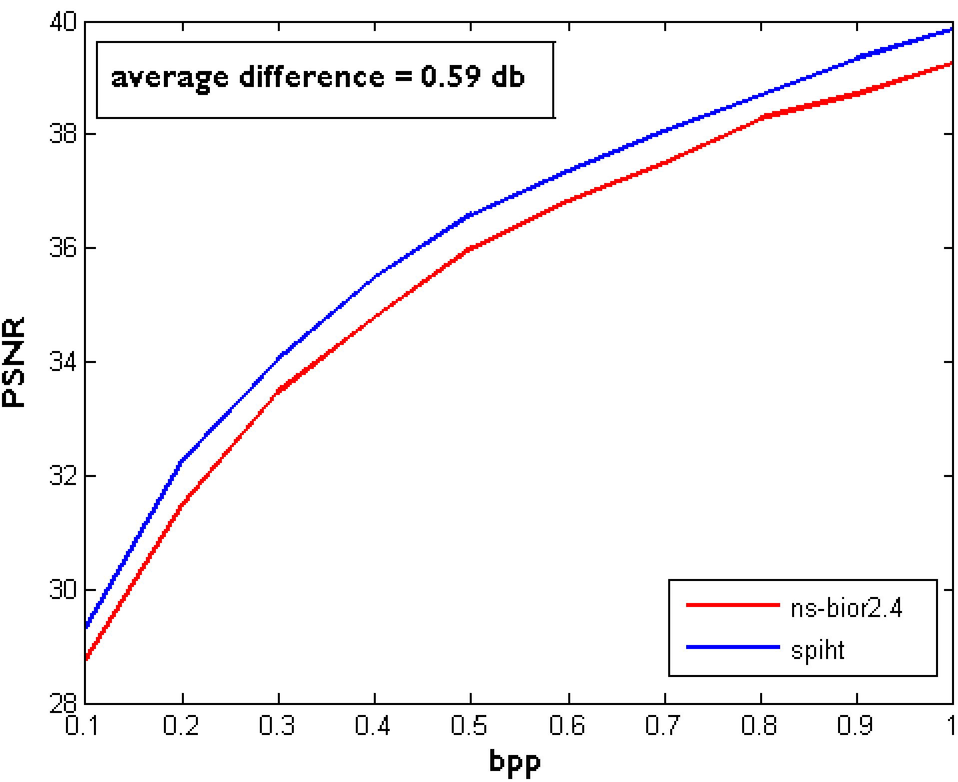


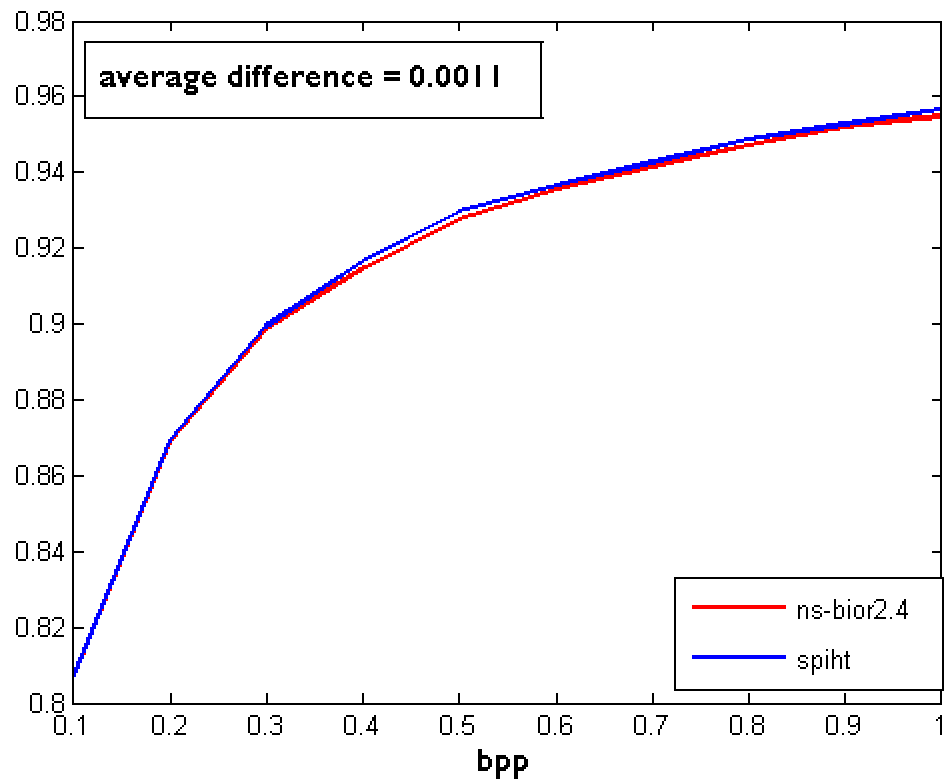


Some applications: image compression

Comparison with SPIHT : **smaller PSNR and comparable visual quality**

Lena image





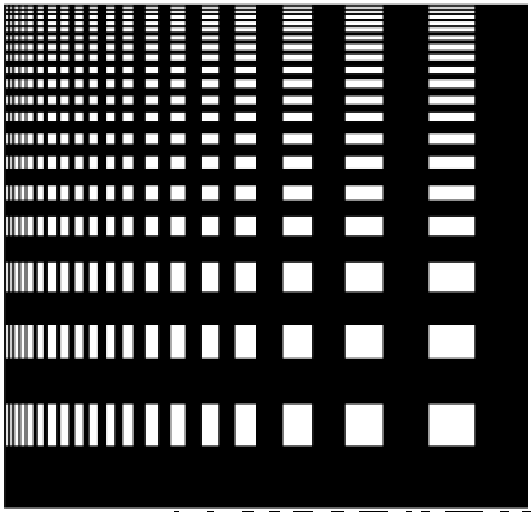
Some applications: edge detection

Ns-filters: better performance than the B-spline filters they converge to

L=3

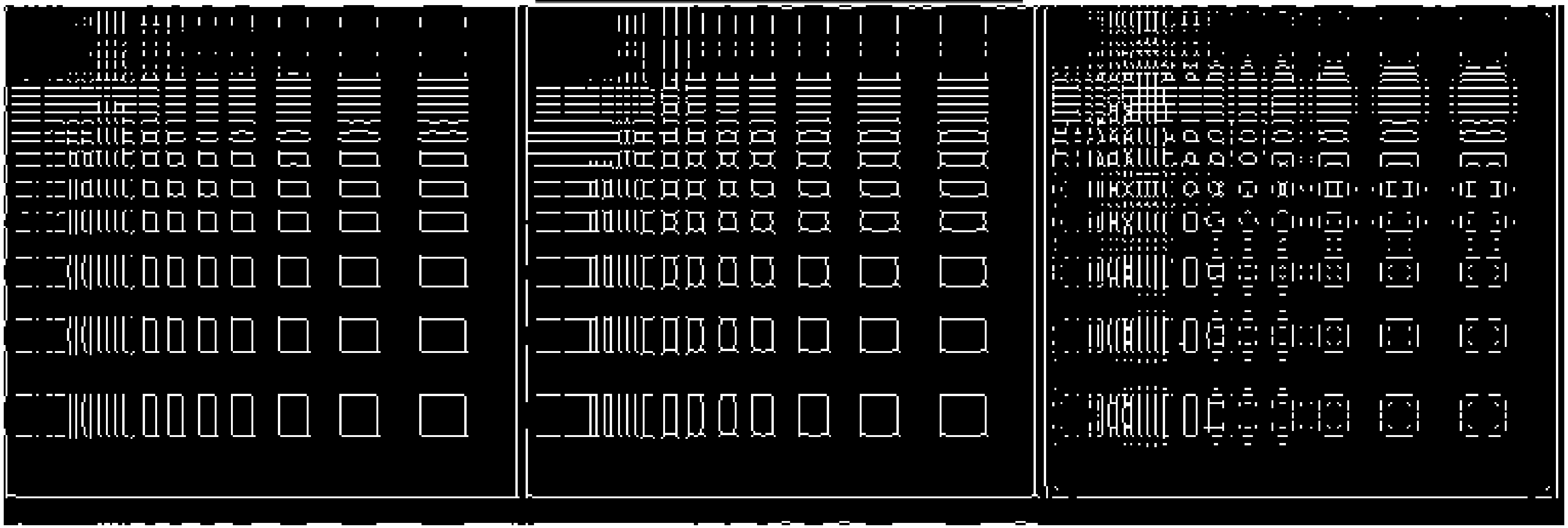
μ=1.6

N_coeff = 5658 (8.5%)



good localization

corner reconstruction



ns-bior1.3 bior1.3 bior3.3

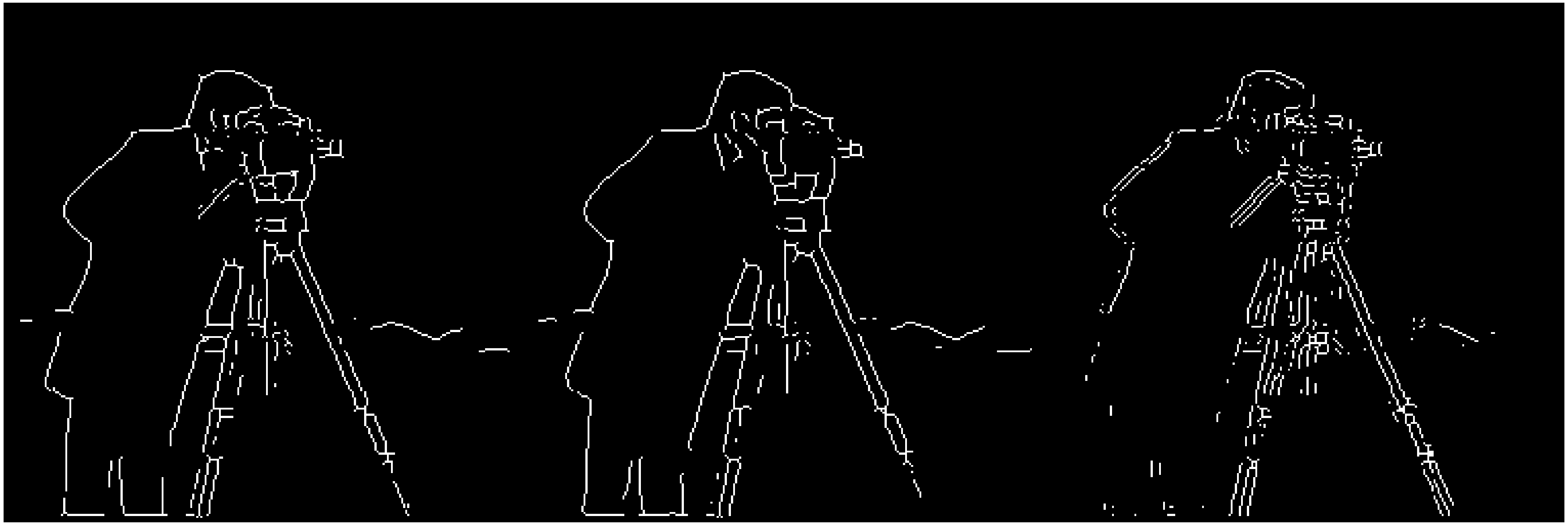
Some applications: edge detection

Ns-filters: better performance than the b-spline filters they converge to

L=3
μ=1.6
N_coeff = 1632 (2.5%)

lack of spurious edges

corners without interruptions



ns-bior1.3 bior1.3 bior3.3

Conclusions

1. NS multiresolution analysis is a general framework for generating new families of wavelets
2. NS wavelets provide more flexibility (support, number of vanishing moments, easy and fast implementation)
3. NS filters are useful in image processing problems:
   - improvement of compaction properties w.r.t. the stationary limit case
   - better visual quality

Future research

- Optimal μ / Adaptive μ
- Interscale relationship in novel algorithms for image processing (compression, denoising, feature extraction, etc.)

SMART 2017

17th – 21th September, 2017 Gaeta, Italy

2nd International Conference on Subdivision; Geometric and Algebraic Methods, Isogeometric Analysis and Refinability in ITaly

www.sbai.uniroma1.it/smart2017

Topics

Topics include Algebraic and Differential Geometry, Computer Aided Design, Curve and Surface Design, Finite Elements, NURBS and Isogeometric Analysis, Refinability, Approximation Theory, Subdivision, Wavelets and Multiresolution Methods….

Organizing Committee

Costanza Conti (Univ. Firenze)
Mariantonia Cotronei (Univ. Reggio Calabria)
Serena Morigi (Univ. Bologna)
Enza Pellegrino (Univ. L'Aquila)
Francesca Pelosi (Univ. Roma "Tor Vergata")
Francesca Pitolli (Univ. Roma "La Sapienza")
Sara Remogna (Univ. Torino)
Lucia Romani (Univ. Milano-Bicocca)
Maria Lucia Sampoli (Univ. Siena)
Alessandra Sestini (Univ. Firenze)

Venue

Venue: Hotel Serapo
Located on the slopes of the Natural Park of Monte Orlando in the most beautiful and panoramic corner overlooking Serapo Beach, very close to the city center and the old town of Gaeta.