Non-Consistent Cell-Average Multiresolution Operators: The Case of the PPH Nonlinear Prediction Operator

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Applied Approximation, Signals, Images,
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Main focus

Construction of a MR prediction in cell-average context which avoids the Gibbs phenomenon in the discontinuities

Review Multiresolution “à la Harten”

- Decimation and prediction operators
- Cell-average and prediction based on polynomial interpolation

Non consistency. Strategy (AY)

- Motivation. Classical strategy (E1)
- Properties of the non-consistent operators

PPH non-linear prediction

- Motivation and definition
- Properties: order, stability and monotonicity
- Numerical examples

Operators

- Let V^k be a discrete space, where k is the level of resolution ($\uparrow k \quad \equiv \quad \uparrow$ resolution).
- Define transfer operators connecting consecutive levels,
  - **Decimation**, $\mathcal{D}_k^{k-1} : V^k \to V^{k-1}$, operator linear and onto
  - **Prediction**, $\mathcal{P}_{k-1}^k : V^{k-1} \to V^k$, no necessary linear
  - $e^k = f^k - \mathcal{P}_{k-1}^k \mathcal{D}_k^{k-1} f^k$

Theorem

$$\mathcal{D}_k^{k-1}\mathcal{P}_{k-1}^k = I_{V^{k-1}} \Longleftrightarrow e^k \in \mathcal{N}(\mathcal{D}_k^{k-1})$$

$\dim \mathcal{N}(\mathcal{D}_k^{k-1}) = \dim V^k - \dim V^{k-1}$
If $\{\mu_j^k\}$ is a basis of $\mathcal{N}(\mathcal{D}_k^{k-1})$, therefore

$$e^k = \sum d_j^k \mu_j^k$$

Define the operators

$$G_k : \begin{array}{ccc} \mathcal{N}(\mathcal{D}_k^{k-1}) & \longrightarrow & \mathcal{G}^k \\ e^k & \longrightarrow & \{d_j^k\} \end{array} \qquad E_k : \begin{array}{ccc} \mathcal{G}^k & \longrightarrow & \mathcal{N}(\mathcal{D}_k^{k-1}) \\ d^k & \longrightarrow & \sum d_j^k \mu_j^k \end{array}$$

Therefore,

$$f^k \stackrel{1:1}{\longleftrightarrow} \{f^{k-1}, d^k\}$$

Multi-scale decomposition

Repeat several times,

$$
\begin{array}{ccccccc}
f^L & \longrightarrow & f^{L-1} & \longrightarrow & f^{L-2} & \longrightarrow & \dots \\
 & \searrow & & \searrow & & \searrow & \\
 & & d^L & & d^{L-1} & & \dots
\end{array}
$$

$$f^L \overset{1:1}{\longleftrightarrow} Mf^L = \{f^0, d^1, \dots, d^L\}$$

Definition (Order of the scheme)

Let p be a polynomial of degree r, i.e., $p(x) \in \Pi^r(\mathbb{R})$; and let p^k be the discretization on the level k. The multiresolution scheme $\{\mathcal{D}_k^{k-1}, \mathcal{P}_{k-1}^k\}$ has order $r+1$ if and only if

$$\mathcal{P}_{k-1}^k \mathcal{D}_k^{k-1} p^k = p^k, \text{ for each resolution level } k.$$

Definition (Stability of MR)

The reconstruction algorithm is stable with respect to the norm $||\cdot||$ if: $\exists$ C such that $\forall$ $j_0 > 0$,

$$\forall\ (f^0, d^1, \ldots, d^{j_0}) \mapsto f^{j_0},$$

$$(\hat{f}^0, \hat{d}^0, \ldots, \hat{d}^{j_0}) \mapsto \hat{f}^{j_0}\ :$$

$$||f^{j_0} - \hat{f}^{j_0}|| \leq C \sup(||f^{j_0-1} - \hat{f}^{j_0-1}||, ||d^{j_0} - \hat{d}^{j_0}||). \qquad (1)$$

Cell-average analysis in 1D

- $X^k = \{x_j^k\}, \quad x_j^k = jh_k$
  $j = 0, \dots, J_k \quad J_k h_k = 1, \quad h_k = 2^{-k} J_0,$
- $c_j^k = [x_{j-1}^k, x_j^k], \quad j = 1, \dots, J_k$

Definition

We define the discretization operator as,

$$f_j^k = \int_{c_j^k} f(x)dx, \quad j = 1, \dots, J_k$$

We define the decimation operator as,

$$(\mathcal{D}_k^{k-1} f^k)_j = \frac{1}{2}(f_{2j-1}^k + f_{2j}^k) = f_j^{k-1}, \quad j = 1, \dots, J_{k-1}$$

Prediction operator in 1D: MR based on polynomial interpolation

- We take $\Pi^r(\mathbb{R}) = \{p(x) = \sum_{0\leq l\leq r} a_l x^l : a_l \in \mathbb{R}\}$
- For each point x_j^{k-1} we have the polynomial,

$$p_j^r(x) = \sum_{l=0}^{r} a_l x^l$$

- The coefficients a_l will be determined with the $r+1$ conditions:

$$f_{j-\frac{r}{2}+l}^{k-1} = \int_{c_{j-\frac{r}{2}+l}^{k-1}} p_j^r(x)dx \quad l = 0, \ldots, r$$

Prediction operator in 1D: MR based on polynomial interpolation

- $r = 0$

$$\begin{cases} (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j-1} = f_j^{k-1} \\ (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} = f_j^{k-1} \end{cases}$$

- $r = 1$ Chaikin's scheme [Non-consistent operator]

$$\begin{cases} (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j-1} = \frac{3}{4} f_j^{k-1} + \frac{1}{4} f_{j-1}^{k-1} \\ (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} = \frac{3}{4} f_j^{k-1} + \frac{1}{4} f_{j+1}^{k-1} \end{cases}$$

- $r = 2$, CA scheme.

$$\begin{cases} (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j-1} = f_j^{k-1} + \frac{1}{8}(f_{j-1}^{k-1} - f_{j+1}^{k-1}) \\ (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} = f_j^{k-1} - \frac{1}{8}(f_{j-1}^{k-1} - f_{j+1}^{k-1}) \end{cases}$$

References

- F. ARÀNDIGA, R. DONAT, A. HARTEN (1998): “Multiresolution based on weighted averages of the hat function I: linear reconstruction technique”, *SIAM J. Numer. Anal., 36*, 160-203
- F. ARÀNDIGA, R. DONAT (2000): “Nonlinear multiscale descompositions: The approach of A. Harten”, *Numerical Algorithms, 23*, 175-216
- A. COHEN (2003): “Numerical Analysis of Wavelet Methods”, *Elsevier*
- A. HARTEN (1993): “Discrete multiresolution analysis and generalized wavelets”, *J. Appl. Numer. Math., 12*, 153–192
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Making the details

We define the errors between f^k and the prediction as:

$$e^k_{2j-1} = f^k_{2j-1} - (\mathcal{P}^k_{k-1} f^{k-1})_{2j-1}, \quad 1 \leq j \leq J_{k-1}$$
$$e^k_{2j} = f^k_{2j} - (\mathcal{P}^k_{k-1} f^{k-1})_{2j}, \quad 1 \leq j \leq J_{k-1}$$

$$f^k \equiv (f^{k-1}, e^k)$$

Motivation

$$(d_1^k)_j = \frac{1}{2}(e^k_{2j-1} - e^k_{2j})$$
$$(d_0^k)_j = \frac{1}{2}(e^k_{2j-1} + e^k_{2j})$$

Motivation

$$(d_1^k)_j = \frac{1}{2}(e_{2j-1}^k - e_{2j}^k)$$
$$(d_0^k)_j = \frac{1}{2}(e_{2j-1}^k + e_{2j}^k)$$

The consistency property is:

$$\mathcal{D}_k^{k-1}\mathcal{P}_{k-1}^k = I_{V^{k-1}} \to \mathcal{D}_k^{k-1}e^k = 0$$

Motivation

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Motivation

$$(d_1^k)_j = \frac{1}{2}(e^k_{2j-1} - e^k_{2j})$$
$$(d_0^k)_j = \frac{1}{2}(e^k_{2j-1} + e^k_{2j})$$

Non-consistency property:

$$\mathcal{D}_k^{k-1}\mathcal{P}_{k-1}^k \neq I_{V^{k-1}} \rightarrow \mathcal{D}_k^{k-1} e^k \neq 0$$

Motivation

$$(d_1^k)_j = \frac{1}{2}(e_{2j-1}^k - e_{2j}^k)$$

$$(d_0^k)_j = f_j^{k-1} - (\mathcal{D}_k^{k-1}\mathcal{P}_{k-1}^k f^{k-1})_j = f_j^{k-1} - \xi_j^{k-1}$$

Non-consistency property:

$$\mathcal{D}_k^{k-1}\mathcal{P}_{k-1}^k \neq I_{V^{k-1}} \to \mathcal{D}_k^{k-1} e^k \neq 0 \to f^k \equiv (f^{k-1}, d_1^k)$$

Strategy (E1)

$$f_{2j}^k = (\mathcal{P}_{k-1}^k f^{k-1})_{2j} + (d_0^k)_j - (d_1^k)_j$$

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$$f_{2j}^{k} = (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} + f_j^{k-1} - \xi_j^{k-1} - (d_1^k)_j$$

Strategy (E1)

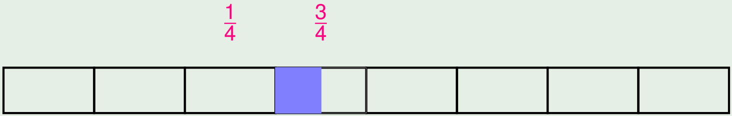
$$f^k_{2j} = \underbrace{(\mathcal{P}^k_{k-1} f^{k-1})_{2j} + f^{k-1}_j - \xi^{k-1}_j}_{(\check{\mathcal{P}}^k_{k-1} f^{k-1})_{2j}} \underbrace{-(d^k_1)_j}_{\check{e}^k_{2j}}$$

Strategy (E1)

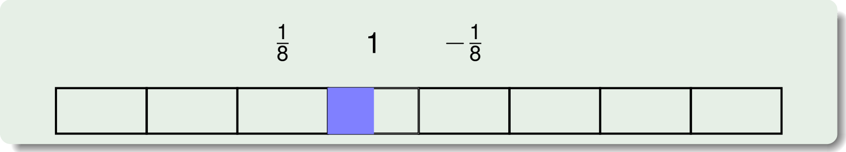
$$f^k_{2j-1} = \underbrace{(\mathcal{P}^k_{k-1} f^{k-1})_{2j-1} + (d^k_0)_j}_{(\check{\mathcal{P}}^k_{k-1} f^{k-1})_{2j-1}} + \underbrace{(d^k_1)_j}_{\check{e}^k_{2j-1}}$$

$$f^k_{2j} = \underbrace{(\mathcal{P}^k_{k-1} f^{k-1})_{2j} + (d^k_0)_j}_{(\check{\mathcal{P}}^k_{k-1} f^{k-1})_{2j}} - \underbrace{(d^k_1)_j}_{\check{e}^k_{2j}}$$

Strategy (E1). Example, $\mathcal{P}_{k-1}^{k}$



Strategy (E1). Example, $\check{\mathcal{P}}_{k-1}^{k}$



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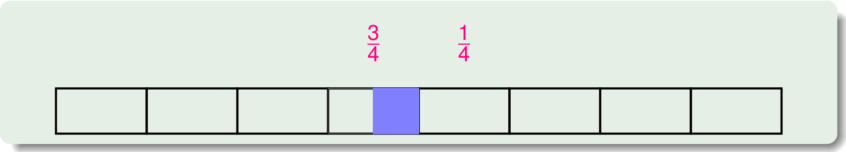

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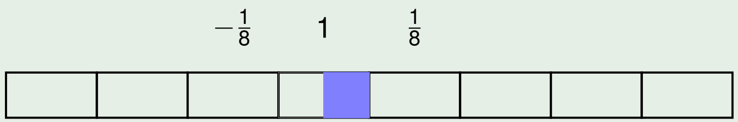

Strategy (AY)

$$f_{2j}^{k} = (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} + (d_0)_j^k - (d_1^k)_{i,j}$$

Strategy (AY)

$$f_{2j}^{k} = (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} + f_j^{k-1} - \xi_j^{k-1} - (d_1^k)_{i,j}$$

Strategy (AY)

$$f_{2j}^k = \underbrace{(\mathcal{P}_{k-1}^k f^{k-1})_{2j}}_{(\mathcal{P}_{k-1}^k f^{k-1})_{2j}} \underbrace{+ f_j^{k-1} - \xi_j^{k-1} - (d_1^k)_j}_{e_{2j}^k}$$

Strategy (AY)

$$f^k_{2j-1} = \underbrace{(\mathcal{P}^k_{k-1} f^{k-1})_{2j-1}}_{(\mathcal{P}^k_{k-1} f^{k-1})_{2j-1}} + \underbrace{(d^k_0)_j + (d^k_1)_j}_{e^k_{2j-1}}$$

$$f^k_{2j} = \underbrace{(\mathcal{P}^k_{k-1} f^{k-1})_{2j}}_{(\mathcal{P}^k_{k-1} f^{k-1})_{2j}} + \underbrace{(d^k_0)_j - (d^k_1)_j}_{e^k_{2j}}$$

Strategy (E1) vs Strategy (AY)

Cell $(2j)$:

$(E1)$

$$f^k_{2j} = \underbrace{(\mathcal{P}^k_{k-1} f^{k-1})_{2j} + f^{k-1}_j - \xi^{k-1}_j}_{(\check{\mathcal{P}}^k_{k-1} f^{k-1})_{2j}} \underbrace{-(d^k_1)_j}_{\check{e}^k_{2i,2j}}$$

(AY)

$$f^k_{2j} = \underbrace{(\mathcal{P}^k_{k-1} f^{k-1})_{2j}}_{(\mathcal{P}^k_{k-1} f^{k-1})_{2j}} + \underbrace{f^{k-1}_j - \xi^{k-1}_j - (d^k_1)_j}_{e^k_{2j}}$$

Strategy (E1) vs Strategy (AY). Algorithm

Inverse transform (E1) 2D

$$
\begin{aligned}
&\varepsilon = (\varepsilon_k) \text{ and } 0 \leq \kappa \leq 1 \\
&\text{for} \quad k = 1, \ldots, L \\
&\text{for} \quad j = 1, \ldots, J_{k-1} \\
&\hat{\xi}_j^{k-1} = (\mathcal{D}_k^{k-1} \mathcal{P}_{k-1}^k \hat{f}^{k-1})_j \\
&(d_0^k)_j = \hat{f}_{i,j}^{k-1} - \hat{\xi}_j^{k-1} \\
&(\hat{d}_0^k)_j = tr((d_0^k)_j, \kappa \varepsilon_{k-1}) \\
&\hat{f}_{2j-1}^k = (\mathcal{P}_{k-1}^k \hat{f}^{k-1})_{2j-1} + (d_0^k)_j + (\hat{d}_1^k)_j \\
&\hat{f}_{2j}^k = (\mathcal{P}_{k-1}^k \hat{f}^{k-1})_{2j-1} + (d_0^k)_j - (\hat{d}_1^k)_j
\end{aligned}
$$

Aràndiga and Yáñez (2016): "Non-consistent cell-average MR operators with application to image processing." *Applied Mathematics and Computation*

Strategy (E1) vs Strategy (AY). Algorithm

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$$\hat{f}_{2j-1}^k = (\mathcal{P}_{k-1}^k \hat{f}^{k-1})_{2j-1} + (\hat{d}_0^k)_j + (\hat{d}_1^k)_j$$
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Motivation

Amat, Dadourian, Liandrat, Ruiz, Trillo (2010): “On a class of L^1-stable nonlinear cell-average MR schemes.” *Journal of Comput. and Applied Mathematics*

$$\begin{cases} (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j-1} = f_j^{k-1} + \frac{1}{8}(f_{j-1}^{k-1} - f_{j+1}^{k-1}) \\ (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} = f_j^{k-1} + \frac{1}{8}(f_{j-1}^{k-1} - f_{j+1}^{k-1}) \end{cases}$$

where $\Delta f_j^k = f_j^k - f_{j-1}^k$ where $M_2(x,y) = \varepsilon_2(x,y)\frac{1}{2}(x+y)$ with

$$\varepsilon_2(x,y) = \left|\frac{sign(x)+sign(y)}{2}\right|\left(1-\left|\frac{x-y}{x+y}\right|^2\right), \forall\, x,y \in \mathbb{R}\setminus\{0\};$$

$$\varepsilon_2(x,0) = 0,\ \forall\, x \in \mathbb{R}; \qquad \varepsilon_2(0,y) = 0,\ \forall\, y \in \mathbb{R}.$$

Serna and Marquina (2004)

Motivation

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$$\begin{cases} (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j-1} = f_j^{k-1} - \frac{1}{4}\left(\frac{\Delta f_j^k + \Delta f_{j+1}^k}{2}\right) \\ (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} = f_j^{k-1} + \frac{1}{4}\left(\frac{\Delta f_j^k + \Delta f_{j+1}^k}{2}\right) \end{cases}$$

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$$\begin{cases} (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j-1} = f_j^{k-1} - \frac{1}{4} M_2(\Delta f_j^k, \Delta f_{j+1}^k) \\ (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} = f_j^{k-1} + \frac{1}{4} M_2(\Delta f_j^k, \Delta f_{j+1}^k) \end{cases}$$

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New family of operators

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New family of operators

$$\begin{cases} (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j-1} = \frac{3}{4} f_j^{k-1} + \frac{1}{4} f_{j-1}^{k-1} - \frac{1}{4}\left(\frac{\Delta f_{j+1}^k - \Delta f_j^k}{2}\right) \\ (\mathcal{P}_{k-1}^{k} f^{k-1})_{2j} = \frac{3}{4} f_j^{k-1} + \frac{1}{4} f_{j+1}^{k-1} - \frac{1}{4}\left(\frac{\Delta f_{j+1}^k - \Delta f_j^k}{2}\right) \end{cases}$$

where $\Delta f_j^k = f_j^k - f_{j-1}^k$ where $D_2(x,y) = \varepsilon_2(x,y)\frac{1}{2}(x-y)$ with

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New family of operators

$$\begin{cases} (\mathcal{P}_{k-1}^k f^{k-1})_{2j-1} = \frac{3}{4} f_j^{k-1} + \frac{1}{4} f_{j-1}^{k-1} - \frac{1}{4} D_2(\Delta f_{j+1}^k, \Delta f_j^k) \\ (\mathcal{P}_{k-1}^k f^{k-1})_{2j} = \frac{3}{4} f_j^{k-1} + \frac{1}{4} f_{j+1}^{k-1} - \frac{1}{4} D_2(\Delta f_{j+1}^k, \Delta f_j^k) \end{cases}$$

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$$\varepsilon_2(x,0) = 0,\ \forall\, x \in \mathbb{R}; \qquad \varepsilon_2(0,y) = 0,\ \forall\, y \in \mathbb{R}.$$

New family of operators: Some properties

Order of approximation

If $S_{\text{WMD}} = \mathcal{P}^k_{k-1}$. For any function $f \in \mathcal{C}^3(\mathbb{R})$, $h > 0$ and $f^0 = \{f(jh)\}_{j\in\mathbb{Z}}$ then
if $\Delta f^{k-1}_{j+1} \Delta f^{k-1}_{j} > 0$ for all $j = 1, \ldots, J_{k-1} - 1$, $k \in \mathbb{N}$,

$$|(S_{\text{WMD}} f^0)_j - f(j\frac{h}{2})| \leq \mathcal{O}(h^3),$$

otherwise

$$|(S_{\text{WMD}} f^0)_j - f(j\frac{h}{2})| \leq \mathcal{O}(h^2),$$

New family of operators: Some properties

Theorem

Harizanov and Oswald (2010): Stability of non-linear and multiscale transforms. Constructive Approximation
Let S_{NL} be a non-linear subdivision scheme defined by:

$$(S_{NL}f)_j = (Sf)_j + F(\delta f)_j,$$

with δ a linear and continuous operator in l^∞ and F a non-linear operator in l^∞. If S_{NL}, F and δ satisfy that

$$\exists M > 0 \quad : \quad \forall d \in l^\infty \quad ||F(d)||_\infty \leq M||d||_\infty \qquad (2)$$

$$\exists L > 0, \exists c < 1 \quad : \quad \forall f \in l^\infty \quad ||\delta S^L_{NL}(f)||_\infty \leq c||\delta(f)||_\infty \qquad (3)$$

then the subdivision scheme S_{NL} is convergent.

New family of operators: Some properties

Convergence of the subdivision scheme

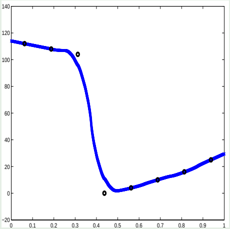
The subdivision scheme S_{WMD} is convergent.

Monotonicity preservation

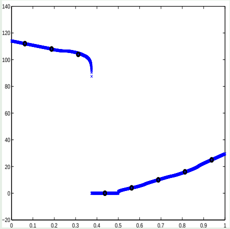
Under certain conditions, the subdivision scheme S_{WMD} preserves the monotony of the points

New family of operators: Some properties

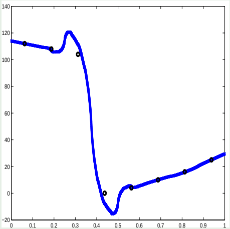
Example $f^0 = (112, 108, 104, 0, 4, 10, 16, 25)$



S_{WMD}



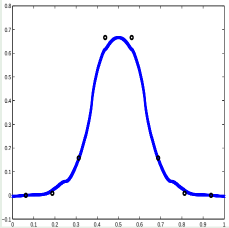
Amat et al.



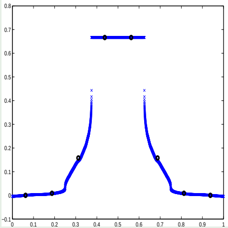
CA

New family of operators: Some properties

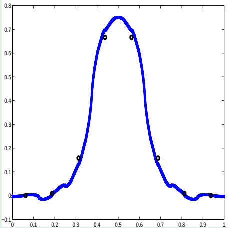
Example $y = [-2.1 : 0.6 : 2.1]$ **and** $f_j^0 = f(y_j)$



S_{WMD}



Amat et al.



CA

New family of operators: Some properties of the MR scheme

Stability of the MR scheme

$$||f^k - \hat{f}^k||_\infty \leq 2||f^{k-1} - \hat{f}^{k-1}||_\infty + ||d_0^k - \hat{d}_0^k||_\infty + ||d_1^k - \hat{d}_1^k||_\infty$$
$$||f^k - \hat{f}^k||_1 \leq 2||f^{k-1} - \hat{f}^{k-1}||_1 + ||d_0^k - \hat{d}_0^k||_1 + ||d_1^k - \hat{d}_1^k||_1$$

Order of the MR scheme

The prediction operator, $\mathcal{P}_{k-1}^k$, reproduces the polynomials of degree 1 in cell-average context.

Some examples

$$(\hat{d}_1^k)_j = \begin{cases} (d_1^k)_j, & \text{if } |(d_1^k)_j| \geq \varepsilon_k; \\ 0, & \text{if } |(d_1^k)_j| < \varepsilon_k; \end{cases}$$

with

$$\varepsilon_{k-1} = \begin{cases} \frac{\varepsilon_k}{2}, & \text{if} \quad \varepsilon_k \geq 1/2; \\ 1/2, & \text{if} \quad \varepsilon_k < 1/2 \end{cases}$$

$$(\hat{d}_0^k)_j = \begin{cases} (d_0^k)_j, & \text{if } |(d_0^k)_j| \geq \kappa\varepsilon_{k-1}; \\ 0, & \text{if } |(d_0^k)_j| < \kappa\varepsilon_{k-1}; \end{cases}$$

with $\kappa = 0.3$.

- PSNR = Peak signal noise ratio $\uparrow$ PSNR $\equiv$ $\uparrow$ quality
- NNZ = Non-zero elements $\uparrow$ NNZ $\equiv$ $\downarrow$ compression

Numerical experiments: Image *lena* with $L = 4$

	$\varepsilon = 8$			$\varepsilon = 16$		
	ℓ_1	PSNR	NNZ	ℓ_1	PSNR	NNZ
CA	3.024	36.10	20085	4.204	32.75	8930
WMD	3.028	36.07	20087	4.195	32.73	8930
PPH	3.108	35.81	20815	4.312	32.40	9303
	$\varepsilon = 32$			$\varepsilon = 40$		
	ℓ_1	PSNR	NNZ	ℓ_1	PSNR	NNZ
CA	5.755	29.58	3616	6.298	28.68	2651
WMD	5.732	29.53	3616	6.285	28.61	2651
PPH	5.865	29.18	3715	6.405	28.31	2745

Numerical experiments: Image *lena* with $L = 4$

Original image

Numerical experiments: Image *lena* with $L = 4$

CA

Numerical experiments: Image *lena* with $L = 4$

WMD

Numerical experiments: Image *lena* with $L = 4$

PPH

Numerical experiments: Image *lena* with $L = 4$

Original image

Numerical experiments: Image *lena* with $L = 4$

CA

Numerical experiments: Image *lena* with $L = 4$

WMD

Numerical experiments: Image *lena* with $L = 4$

PPH

Numerical experiments: Image *barbara* with $L = 4$

	$\varepsilon = 8$			$\varepsilon = 16$		
	ℓ_1	PSNR	NNZ	ℓ_1	PSNR	NNZ
CA	3.482	34.95	49152	5.739	30.16	26454
WMD	3.488	34.92	49153	5.746	30.11	26456
PPH	3.573	34.64	52316	5.955	29.69	27615
	$\varepsilon = 32$			$\varepsilon = 40$		
	ℓ_1	PSNR	NNZ	ℓ_1	PSNR	NNZ
CA	8.857	26.14	10888	10.107	24.96	6968
WMD	8.895	26.07	10888	10.154	24.90	6968
PPH	9.111	25.71	10654	10.308	24.59	6731

Numerical experiments: Image *geopa* with $L = 4$

	$\varepsilon = 8$			$\varepsilon = 16$		
	ℓ_1	PSNR	NNZ	ℓ_1	PSNR	NNZ
CA	0.477	45.37	6685	1.284	38.32	3136
WMD	0.396	45.73	6685	1.093	38.70	3136
PPH	0.117	48.57	3521	0.372	40.97	2149
	$\varepsilon = 32$			$\varepsilon = 40$		
	ℓ_1	PSNR	NNZ	ℓ_1	PSNR	NNZ
CA	2.043	34.83	1679	2.410	33.43	1261
WMD	1.712	35.35	1679	2.081	33.82	1261
PPH	0.713	36.78	1226	0.868	35.38	989

Present work

- Prove the stability in ℓ^2
- Use the schemes in other applications as signal denoising
- Generalize the method using a scheme with a major number of cells
- Introduce properties to generalize the function D_2

Thank you!

Present work

- Prove the stability in ℓ^2
- Use the schemes in other applications as signal denoising
- Generalize the method using a scheme with a major number of cells
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Thank you!