MULTIGRID METHODS
and
SUBDIVISION SCHEMES

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MGM:

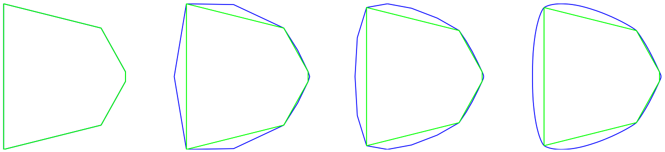
- iterative method for solving linear systems

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n}, \qquad b \in \mathbb{C}^{n}, \qquad n \in \mathbb{N},$$

  with A circulant, symmetric and positive definite.
- Applications: elliptic PDEs

Subdivision:

- iterative method for generation of curves and surfaces.
- Applications: computer animation



Results of application of subdivision schemes to multigrid methods

Subdivision scheme	$n = 242$		$n = 728$		$n = 2187$	
	iter	converg. rate	iter	converg. rate	iter	converg. rate
Linear Bspline	86	0.829	214	0.9273	476	0.9667
Interp. 3-point	66	0.7820	101	0.8519	137	0.8887
Cubic Bspline	29	0.5729	52	0.733	71	0.7956
Interp. 4-point	42	0.6784	45	0.6955	46	0.703

Table: MGM for univariate biharmonic problem

Elliptic PDEs with periodic boundary conditions (PBC)

For $q \in \mathbb{N}$ and $w \in C^0\big([a,b]\big)$, find $u \in C^{2q}\big([a,b]\big)$ such that:

$$\begin{cases} (-1)^q\, u^{(2q)}(x) = w(x) & x \in (a,b), \\ u^{(m)}(a) = u^{(m)}(b) = h_m \in \mathbb{R} & m = 0,\ldots,2q-1. \end{cases}$$

Lapacian problem (q=1):

- Discretization with finite differences $\Longrightarrow$

$$A_n = \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{bmatrix}_{n\times n} \quad \text{is} \quad \begin{cases} \text{circulant} \\ \text{symmetric} \\ \text{positive definite} \end{cases}$$

- $w(x)$, boundary conditions $\Longrightarrow$ $b_n \in \mathbb{C}^n$

First step of Two Grid Method: Smoother

Solve $A_n x = b_n$ with $A_n = \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{bmatrix}$.

Weighted Jacobi: for $0 < \omega \leq 1$ and D_n the main diagonal of A_n

$$x^{\ell+1} := x^\ell + \omega D_n^{-1}(b_n - A_n x^\ell) = \underbrace{\left(I_n - \frac{\omega}{2} A_n\right)}_{:=J_{\omega,n}} x^\ell + \underbrace{\frac{\omega}{2} b_n}_{:=d_n}, \quad \ell \in \mathbb{N}_0.$$

Eigenvalues λ_k and Eigenvectors $\boldsymbol{v}_k$ of $J_{\omega,n}$ for $k = 1, \ldots, n$:

$$\lambda_k = 1 - 2\omega \sin^2\left(\frac{k\pi}{2(n+1)}\right), \quad \boldsymbol{v}_k = \sin(k\boldsymbol{v}), \quad \boldsymbol{v} = \left[\frac{j\pi}{n+1}\right]_{j=1}^{n}.$$

Error: $\boldsymbol{e} = \sum_{k=1}^{n} c_k \boldsymbol{v}_k, \quad c_k \in \mathbb{R} \quad \Longrightarrow \quad (J_{\omega,n})^\ell \boldsymbol{e} = \sum_{k=1}^{n} c_k \lambda_k^\ell \boldsymbol{v}_k$

For $0 < \omega \leq 1$:

- $|\lambda_k| < 1,\ k = 1, \ldots, n \implies$ weighted Jacobi is convergent
- $\lambda_1 = 1 - O\left(\frac{1}{n^2}\right) \approx 1 \implies$ weighted Jacobi converges slowly
- no value of ω damps out low frequencies

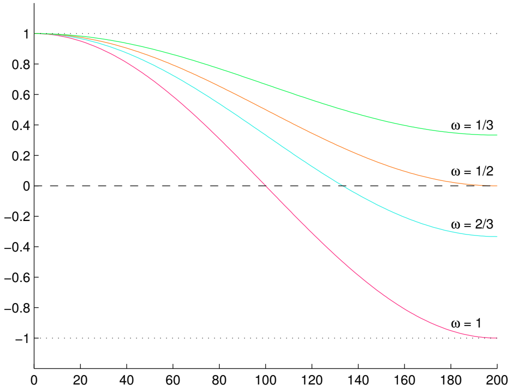


Figure: Plot of eigenvalues λ_k, $k = 1, \ldots, 200$

Exact solution:

$$\bar{x}_i = \sin\left(3\,\frac{\pi(i-1)}{200-1}\right) + \sin\left(150\,\frac{\pi(i-1)}{200-1}\right), \quad i = 1, \ldots, 200.$$

Weighted Jacobi: $x^{\ell+1} = J_{\omega,200}x^{\ell} + d_{200}, \quad \omega = \frac{2}{3}, \quad x^0 = \mathbf{0}.$

Error: $e_i^{\ell} = \bar{x}_i - x_i^{\ell}, \quad i = 1, \ldots, 200$

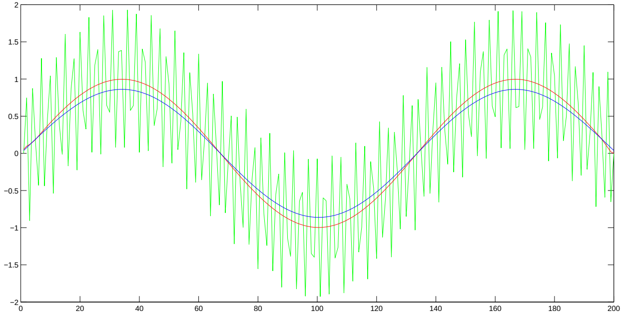


Figure: Error for $\ell = 0$ (green), $\ell = 4$ (red) and $\ell = 200$ (blue)

Second step of Two Grid Method: Coarse Grid Correction

Error equation

$$A_n x = b_n \quad (1) \quad \Longleftrightarrow \quad A_n e = r_n \quad (2)$$

Ingredients:

- $\bar{x}$: exact solution of (1)
- $\tilde{x}$: approximation of $\bar{x}$
- $e = \bar{x} - \tilde{x}$: error
- $r_n = b_n - A_n\tilde{x}$: residual

Idea:

$$\text{Given } \tilde{x} \quad \Longrightarrow \quad \begin{aligned} r_n &= b_n - A_n\tilde{x} \\ e &= \text{ exact solution of (2)} \\ \bar{x} &= \tilde{x} + e \end{aligned}$$

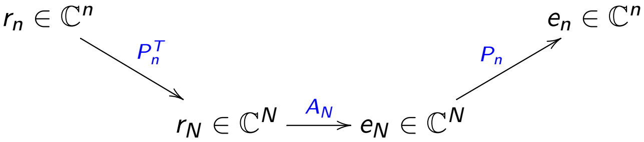
Coarse Grid Correction:

- Input: $\tilde{x}$, $P_n \in \mathbb{C}^{n \times N}$, $N < n$, full-rank matrix
- Compute:

$$\begin{array}{lrcl}
1. & r_n & = & b_n - A_n \tilde{x} \\
2. & r_N & = & P_n^T r_n \\
3. & A_N & = & P_n^T A_n P_n \\
4. & e_N & = & A_N^{-1} r_N \\
5. & e_n & = & P_n e_N \\
6. & \bar{x} & = & \tilde{x} + e_n
\end{array}$$

- Output: $\bar{x}$

Iteration matrix: $CGC = I_n - P_n \left(P_n^T A_n P_n \right)^{-1} P_n^T A_n$



Let

- $\|\cdot\|_{A_n} = \|A_n^{1/2}\cdot\|_2$ and D_n the main diagonal of A_n, $n \in \mathbb{N}$,
- $TGM = \left(J_{\omega,n}\right)^{\ell}\left[I_n - P_n\left(P_n^T A_n P_n\right)^{-1} P_n^T A_n\right], \quad \ell \in \mathbb{N}_0, \quad 0 < \omega \leq 1.$

Theorem (J. W. Ruge and K. Stuben,1987)

If

1. $\exists\alpha > 0$ *independent of n such that*

$$\|J_{\omega,n}x\|_{A_n}^2 \leq \|x\|_{A_n}^2 - \alpha\|x\|_{A_n D_n^{-1} A_n}^2, \qquad \forall x \in \mathbb{C}^n,$$

2. $\exists\beta > 0$ *independent of n such that*

$$\min_{y\in\mathbb{C}^N}\|x - P_n y\|_{D_n}^2 \leq \beta\|x\|_{A_n}^2, \qquad \forall x \in \mathbb{C}^n,$$

then $\beta > \alpha$ *and* $\|TGM\|_{A_n} \leq \sqrt{1 - \frac{\alpha}{\beta}} < 1.$

Structure of iteration matrix: construction of A_n

(another perspective)

Univariate elliptic problems

- Problem: for $q \in \mathbb{N}$

$$\begin{cases} (-1)^q\, u^{(2q)}(x) = w(x) & x \in (a,b), \\ u^{(m)}(a) = u^{(m)}(b) = h_m & m = 0, \ldots, 2q-1. \end{cases}$$

- Approximation: centered finite differences of order $2q$
- Discretization matrix: A_n circulant matrix with symbol f, where

$$f(z) = \left(-\frac{(z-1)^2}{z} \right)^q, \qquad z \in \mathbb{C},\ |z| = 1,$$

and f vanishes at 1 with order $2q$.

$$A_n = C_n(f) \text{ circulant matrix with symbol } f$$

- Circulant matrix: $C_n(f) = F_n D_n(f) F_n^H$, where

$$\begin{aligned} 1.\quad D_n(f) &= \underset{r=0,\ldots,n-1}{\text{diag}} f\left(e^{-\mathrm{i}\frac{2\pi r}{n}}\right), \\ 2.\quad F_n &= \frac{1}{\sqrt{n}}\left[e^{-\mathrm{i}\frac{2\pi rs}{n}}\right]_{r,s=0}^{n-1}. \end{aligned}$$

- Example (non zero coefficients of f):

$$\begin{array}{llll} 1. & \text{Laplacian} & (q=1): & \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \\ 2. & \text{Biharmonic} & (q=2): & \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \end{bmatrix} \end{array}$$

Structure of iteration matrix: construction of P_n

Subdivision schemes of arity $g \geq 2$

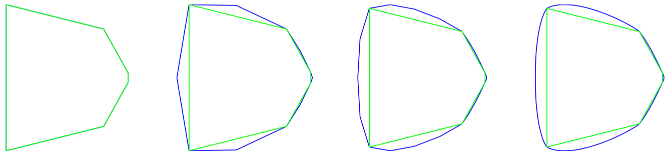
Subdivision Operator: $S_{\mathbf{p}} : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ such that $\forall\, \mathbf{c} \in \ell(\mathbb{Z})$

$$(S_{\mathbf{p}}\mathbf{c})_\alpha := \sum_{\beta \in \mathbb{Z}} p_{\alpha - g\beta} c_\beta, \qquad p_{-\eta} = p_\eta \quad \forall\, \eta \in \mathbb{Z}.$$

Subdivision Scheme: for $\mathbf{c}^0 \in \ell(\mathbb{Z})$, $\quad \mathbf{c}^{\ell+1} := S_{\mathbf{p}}\mathbf{c}^\ell, \quad \ell \in \mathbb{N}_0$.

Symbol: $p(z) = p_0 + \sum_{\alpha \in \mathbb{N}} p_\alpha (z^{-\alpha} + z^\alpha), \quad z \in \mathbb{C}, \quad |z| = 1.$

Convergence:



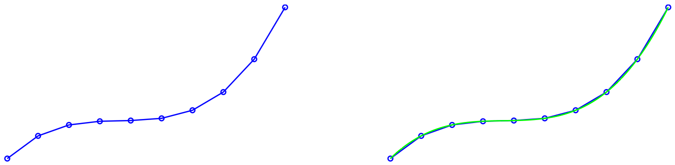


Figure: Example of generation of cubic polynomials

Let denote Π_d the space of polynomials of degree d.

Generation: A convergent subdivision scheme $S_{\mathbf{p}}$ generates polynomials up to degree d_G if

$$\forall\, \mathbf{c} = \{\, \pi(\alpha) \,:\, \alpha \in \mathbb{Z},\, \pi \in \Pi_{d_G} \,\}\,, \qquad \lim_{\ell \to \infty} S_{\mathbf{p}}^{\ell} \mathbf{c} \in \Pi_{d_G}\,.$$

Let $E_g := \left\{ e^{-\mathrm{i}\frac{2\pi j}{g}} \;:\; j = 0, \ldots, g-1 \right\}$. A convergent subdivision scheme $S_{\mathbf{p}}$ generates polynomials up to degree d_G if and only if

$$p(1) = g \quad \text{and} \quad D^j p(\varepsilon) = 0, \quad \forall\, \varepsilon \in E_g \setminus \{\, 1 \,\}\,, \quad j = 0, \ldots, d_G\,.$$

1. Given: $n = g^k$, $g, k \geq 2$ integers, $N = g^{k-1}$
2. Select: p symbol of a certain subdivision scheme of arity g
3. Define: $P_n(p) = C_n(p)K_{n,g}^T \in \mathbb{C}^{n \times N}$, where

$$K_{n,g} = \begin{bmatrix} 1 & 0_{g-1} & & & & & & \\ & & 1 & 0_{g-1} & & & & \\ & & & & \ddots & & & \\ & & & & & & 1 & 0_{g-1} \end{bmatrix} \in \mathbb{C}^{N \times n}.$$

Let $E_g := \left\{ e^{-\mathrm{i}\frac{2\pi j}{g}} \; : \; j = 0, \dots, g-1 \right\}$.

Theorem (M. Bolten, M. Donatelli, T. Huckle and C. Kravvaritis, 2014)

Let f be a Laurent polynomial s.t. $f(z) = 0 \iff z = 1$.
If the symbol p of a subdivision scheme of arity g satisfies

$$
\begin{aligned}
&i) \quad \lim_{z\to 1} \frac{|p(\varepsilon z)|^2}{f(z)} < +\infty \qquad \forall \varepsilon \in E_g \setminus \{1\}, \\
&ii) \quad \sum_{\varepsilon \in E_g} |p(\varepsilon z)|^2 > 0 \qquad \forall z \in \mathbb{C}, \; |z| = 1,
\end{aligned}
$$

then $\exists\, \beta > 0$ independent of n such that

$$\min_{y \in \mathbb{C}^N} \|x - P_n(p) y\|_{D_n}^2 \le \beta \|x\|_{A_n}^2, \qquad \forall x \in \mathbb{C}^n.$$

Let $E_g := \left\{ e^{-\mathrm{i}\frac{2\pi j}{g}} \ : j = 0, \ldots, g-1 \right\}$.

Theorem (M. Donatelli and V. Turati, 2016)

Let f be a Laurent polynomial s.t. $f(z) = 0 \iff z = 1$.
If the symbol p of a subdivision scheme of arity g satisfies

$$i) \quad \lim_{z\to 1} \frac{|p(\varepsilon z)|^2}{f(z)} \ < \ +\infty \qquad \forall \varepsilon \in E_g \setminus \{1\},$$

$$ii) \quad |p(1)|^2 > 0$$

then $\exists\, \beta > 0$ independent of n such that

$$\min_{y\in\mathbb{C}^N} \|x - P_n(p)y\|^2_{D_n} \le \beta \|x\|^2_{A_n}, \qquad \forall x \in \mathbb{C}^n.$$

Let $E_g := \left\{ e^{-\mathrm{i}\frac{2\pi j}{g}} \ : j = 0, \dots, g-1 \right\}$.

Theorem (M. Charina, M. Donatelli, L. Romani and V. Turati, 2016)

Let f be a Laurent polynomial s.t.

$$D^j f(z) = 0 \quad j = 0, \dots, m-1 \qquad \Longleftrightarrow \qquad z = 1.$$

If the symbol p of a subdivision scheme of arity g satisfies

ii) $p(1) = g$

i) $D^j p(\varepsilon) = 0 \quad j = 0, \dots, q, \quad q \geq \lceil \frac{m}{2} \rceil - 1, \quad \forall \varepsilon \in E_g \setminus \{1\}$

then $\exists \beta > 0$ independent of n such that

$$\min_{y \in \mathbb{C}^N} \|x - P_n(p) y\|^2_{D_n} \leq \beta \|x\|^2_{A_n}, \qquad \forall x \in \mathbb{C}^n.$$

Remark: If a convergent subdivision scheme S_p of arity g generates polynomials up to degree q with $q \geq \lceil \frac{m}{2} \rceil - 1$, then the two grid method is convergent and optimal.

TGM Results

Univariate biharmonic problem

Binary subdivision schemes:

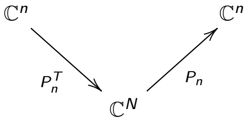
Subdivision scheme	iterations $n = 2^{10}$	$n = 2^{11}$	$n = 2^{12}$	gener. degree	repr. degree	smooth.
Linear Bspline	25	25	25	1	1	C^0
Pseudo $B_{3,1}$	12	12	12	5	3	C^3
Cubic Bspline	11	11	11	3	1	C^2
Interp. 4 point	11	11	11	3	3	C^1
Interp. 6 point	11	11	11	5	5	C^1

Ternary subdivision schemes

Subdivision scheme	iterations $n = 3^6$	$n = 3^7$	$n = 3^8$	gener. degree	repr. degree	smooth.
Linear Bspline	34	34	34	1	1	C^0
Interp. 3-point	32	32	32	1	1	C^1
Cubic Bspline	31	31	31	3	1	C^2
Interp. 4-point	29	29	29	3	3	C^1

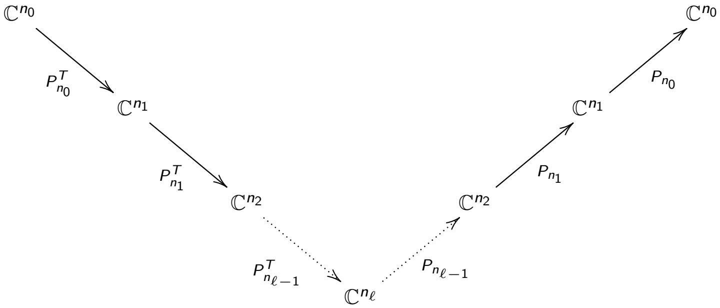
V-cycle method

Let $g, k \in \mathbb{N}$ and $g, k \geq 2$.
TGM: $n = g^k, \quad N = g^{k-1}, \quad P_n(p) = C_n(p)\, K_n^T \in \mathbb{C}^{n \times N}$



V-cycle: let $\ell \in \mathbb{N}$, $1 \leq \ell \leq k-1$, define for $j = 0, \ldots, \ell - 1$

$$n_j = g^{k-j}, \qquad n_\ell = g^{k-\ell}, \qquad P_{n_j}(p) = C_{n_j}(p)\, K_{n_j}^T \in \mathbb{C}^{n_j \times n_{j+1}}$$



- For $j = 0, \ldots, \ell - 1$

$$CGC_j = I_{n_j} - P_{n_j}\left(P_{n_j}^T A_{n_j} P_{n_j}\right)^{-1} P_{n_j}^T A_{n_j} \in \mathbb{C}^{n_j \times n_j}.$$

- For $j = \ell - 1, \ldots, 0$, $m \in \mathbb{N}_0$, $0 < \omega \leq 1$

$$MGM_\ell = O \in \mathbb{C}^{n_\ell \times n_\ell},$$

$$MGM_j = \left(J_{\omega, n_j}\right)^m \left[I_{n_j} - P_{n_j}(I_{n_{j+1}} - MGM_{j+1})\left(P_{n_j}^T A_{n_j} P_{n_j}\right)^{-1} P_{n_j}^T A_{n_j}\right].$$

Theorem (J. W. Ruge and K. Stuben,1987)

If for $j = 0, \ldots, \ell - 1$

1. $\exists \alpha_j > 0$ *independent of n such that*

$$\|J_{\omega, n_j} x\|_{A_{n_j}}^2 \leq \|x\|_{A_{n_j}}^2 - \alpha_j \|x\|_{A_{n_j}^2}^2, \qquad \forall x_{n_j} \in \mathbb{C}^{n_j},$$

2. $\exists \beta_j > 0$ *independent of n such that*

$$\|CGC_j x\|_{A_{n_j}}^2 \leq \beta_j \|x\|_{A_{n_j}^2}^2, \qquad \forall x_{n_j} \in \mathbb{C}^{n_j},$$

then $\delta = \min_j \frac{\alpha_j}{\beta_j} \in (0, 1)$ *and* $\|MGM_0\|_{A_{n_0}} \leq \sqrt{1 - \delta} < 1.$

Theorem (A. Aricò, M. Donatelli, C. Serra-Capizzano, 2004)

Let f be a Laurent polynomial s.t. $f(z) = 0 \iff z = 1$.
If the symbol p of a subdivision scheme of arity 2 satisfies

$$i) \quad \lim_{z \to 1} \frac{|p(-z)|}{f(z)} < +\infty$$

$$ii) \quad |p(z)|^2 + |p(-z)|^2 > 0 \qquad \forall\, z \in \mathbb{C},\ |z| = 1,$$

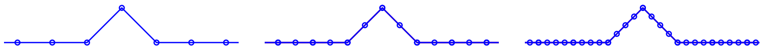
then the approximation property holds.

Remark: For $g \geq 2$,

$$E_g := \left\{ e^{-\mathrm{i}\frac{2\pi j}{g}} \; : \; j = 0, \ldots, g-1 \right\} \implies E_2 = \{ -1, 1 \}$$

ℓ^∞-stability

Basic function ϕ: limit function of a convergent subdivision scheme S_p for initial data $\boldsymbol{\delta} = \{ \delta_{\alpha,0} \ : \ \alpha \in \mathbb{Z} \}$.



ℓ^∞-stability: S_p is ℓ^∞ stable if $\exists\, 0 < A \leq B < \infty$ such that $\forall\, \mathbf{c} \in \ell^\infty(\mathbb{Z})$

$$A \|\mathbf{c}\|_{\ell^\infty(\mathbb{Z})} \leq \left\| \sum_{\alpha \in \mathbb{Z}} c_\alpha \phi(\cdot - \alpha) \right\|_{L^\infty(\mathbb{R})} \leq B \|\mathbf{c}\|_{\ell^\infty(\mathbb{Z})}.$$

Cohen's conditions [1990]: The symbol p of a convergent subdivision scheme $S_{\mathbf{p}}$ satisfies Cohen's condition if

$$\left| p\left(e^{-\mathrm{i}\xi 2^{-k}} \right) \right| > 0, \qquad \forall\, \xi \in [-\pi, \pi], \quad \forall\, k \geq 1.$$

Theorem (M. Charina, M. Donatelli, L. Romani and V. Turati, 2016)

Let f be a Laurent polynomial s.t.

$$D^j f(z) = 0 \quad j = 0, \ldots, m-1 \qquad \Longleftrightarrow \qquad z = 1.$$

If a convergent subdivision scheme of arity 2

i) *generates polynomials up to degree q with $q \geq m-1$,*
ii) *its basic function ϕ is ℓ^∞-stable,*

then the approximation property holds.

Theorem (M. Charina, M. Donatelli, L. Romani and V. Turati, 2016)

Let f be a Laurent polynomial s.t.

$$D^j f(z) = 0 \quad j = 0, \dots, m-1 \qquad \Longleftrightarrow \qquad z = 1.$$

If the symbol p of a subdivision scheme of arity 2 satisfies

$$\begin{array}{lll} i) & D^j p(-1) = 0 & j = 0, \dots, q, \quad q \geq m-1 \\ ii) & \left| p\left(e^{-\mathrm{i}\xi 2^{-k}}\right)\right| > 0, & \forall \xi \in [-\pi, \pi], \quad \forall k \geq 1 \end{array}$$

then the approximation property holds.

V-cycle Results

Binary subdivision schemes

Coarsest grid dimension: 2×2

Subdivision scheme	iterations $n = 2^{10}$	time	$n = 2^{11}$	time	$n = 2^{12}$	time
Linear Bspline	617	1.2205	744	1.7046	801	2.8182
Pseudo $B_{3,1}$	19	0.0961	22	0.1816	24	0.6990
Cubic Bspline	40	0.1201	43	0.2279	45	0.7017
Interp. 4 point	19	0.0917	23	0.1839	26	0.5860
Interp. 6 point	13	0.0785	13	0.1707	14	0.5630

Subdivision scheme	generation degree	reproduction degree	smoothness
Linear Bspline	1	1	C^0
Pseudo $B_{3,1}$	5	3	C^3
Cubic Bspline	3	1	C^2
Interp. 4 point	3	3	C^1
Interp. 6 point	5	5	C^1

Ternary subdivision schemes

Coarsest grid dimension: 3×3

Subdivision scheme	iterations $n = 3^6$	time	$n = 3^7$	time	$n = 3^8$	time
Linear Bspline	462	0.1987	864	0.6125	1057	2.2994
Interp. 3-point	133	0.0675	154	0.1660	169	0.8704
Cubic Bspline	67	0.0483	80	0.1205	87	0.7190
Interp. 4-point	46	0.0386	47	0.0899	53	0.6566

Subdivision scheme	generation degree	reproduction degree	smoothness
Linear Bspline	1	1	C^0
Interp. 3-point	1	1	C^1
Cubic Bspline	3	1	C^2
Interp. 4-point	3	3	C^1

Summary: TGM and V-cycle method

Construction of $A_n(f)$

Let f be a Laurent polynomial s.t.

$$D^j f(z) = 0 \quad j = 0, \ldots, m-1 \qquad \Longleftrightarrow \qquad z = 1.$$

Construction of $P_n(p)$

TGM: If a convergent subdivision scheme S_p of arity g generates polynomials up to degree q with $q \geq \lceil \frac{m}{2} \rceil - 1$, then the two grid method is convergent and optimal.

V-cycle: If a convergent subdivision scheme S_p of arity 2 generates polynomials up to degree q with $q \geq m-1$ and it is ℓ^∞-stable, then the V-cycle method is convergent and optimal.

Thank you for your kind attention...