From Box-splines to four-directional pseudo-splines

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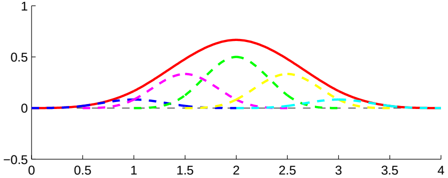
B-splines with uniform knots

It is well known that

- polynomial B-splines with integer knots are refinable functions
- any polynomial spline can be obtained as the limit of a subdivision process



Coefficients of the subdivision mask: $\frac{1}{8},\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{2},\ \frac{1}{8}$

$$B^3(t) = \frac{1}{8}B^3(2t) + \frac{1}{2}B^3(2t-1) + \frac{3}{4}B^3(2t-2) + \frac{1}{2}B^3(2t-3) + \frac{1}{8}B^3(2t-4)$$

The binary subdivision scheme for degree n splines

The polynomial spline $s(t) = \sum_{i \in \mathbb{Z}} P_i B_n(t-i)$ can be obtained via the iterative computation of denser and denser sequence of points (*subdivision scheme*)

$$\mathbf{P}^{(k+1)} = S_{\mathbf{a}^n} \mathbf{P}^{(k)}, \quad P_i^{(k+1)} = \sum_{j \in \mathbb{Z}} a^n_{i-2j} P_j^{(k)}, \quad a_i^n = \frac{1}{2^n} \binom{n+1}{i}, \; i = 0, \cdots, n$$

with the initial sequence of points $\mathbf{P}^{(0)} = \{P_i, \; i \in \mathbb{Z}\}$.

The action of the subdivision operator $S_{\mathbf{a}^n}$ consists in two different rules according to the parity of the indices

$$P_{2i}^{(k+1)} = \sum_{j \in \mathbb{Z}} a^n_{2j} P_{i-j}^{(k)}, \quad P_{2i+1}^{(k+1)} = \sum_{j \in \mathbb{Z}} a^n_{2j+1} P_{i-j}^{(k)},$$

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☞ Any degree n B-spline is the *basic limit function of the corresponding subdivision scheme* when starting with the sequence $\delta = 0, \; 0, \; 1, \; 0, \; 0$.

Binary subdivision scheme

This subdivision idea allows us to define other type of refinable functions not necessarily piecewise polynomial but with useful properties. They find application in several context and, with some extent, provide "generalization" of B-splines.

$$\begin{cases} \text{Input} \quad \mathbf{P}^{(0)} \\ \text{For} \quad k = 0, 1, \cdots \\ \qquad\qquad \mathbf{P}^{(k+1)} := S_{\mathbf{a}} \mathbf{f}^{(k)} \end{cases}$$

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☞ Mostly, this function is *not defined analytically*.

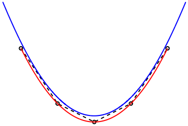
Polynomial generation versus polynomial reproduction

Beside convergence, two important properties of subdivision (also of B-splines subdivision) are

- polynomial generation: the capability of subdivision to provide polynomials in the limit
- polynomial reproduction: the capability of subdivision schemes to provide in the limit exactly the same polynomials from which the data is sampled.

Definition of polynomial generation/reproduction for sub.

To understand the difference between polynomial generation and reproduction we consider a simple example: let $\mathbf{P}^{(0)}$ be sampled from a parabola



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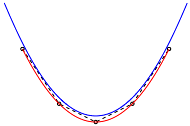

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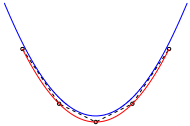

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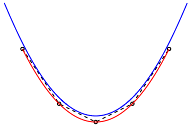

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☞Essentially, the higher is the *number of polynomials reproduced*, the higher is the *approximation order* and the possible *regularity* of the scheme.

☞ But, in contrast, the higher is the number of polynomials reproduced, the *bigger is the size of the support* of the basic limit functions.

Polynomial generation versus polynomial reproduction

Polynomial generation and reproduction of any subdivision scheme can be easily checked using simple algebraic conditions on the subdivision symbol

$$a(z) = \sum_{i \in \mathbb{Z}} a_i\, z^i, \qquad z \in \mathbb{C} \setminus \{0\}.$$

For stable (zero limit only for zero initial sequences) subdivisions schemes,

- generation of polynomials is equivalent to the zero conditions (up to a certain order) of the subdivision symbol
- reproduction of polynomials is equivalent to generation plus additional algebraic conditions on the subdivision symbol.

Algebraic properties for polynomial generation

Proposition [Dyn et al. 2008, C. and Hormann 2011]

- A stable subdivision scheme $S_{\mathbf{a}}$ generates Π_d iff
$a(1) = 2, \quad \left.\frac{d^r\, a(z)}{dz^r}\right|_{z=-1} = 0,\ r = 0, ..., d.$
- A stable subdivision scheme$S_{\mathbf{a}}$ reproduces Π_d with respect to the parametrization $t_i^{(k)} = \frac{i+p}{2^k}$, iff $\left.\frac{d^r\, a(z)}{dz^r}\right|_{z=-1} = 0,\ r = 0, ..., d$ and

$$\left.\frac{d^r\, a(z)}{dz^r}\right|_{z=1} = 2\prod_{j=0}^{r-1}(p-j), \quad r = 1, ..., d.$$

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☞ The previous result allow us to identify the correct parametrization: $p = \frac{1}{2}\frac{d^1\, a(z)}{dz^r}|_{z=1}$. The scheme is primal if $p \in \mathbb{Z}$ and dual if $p \in \frac{1}{2}\mathbb{Z}$.

Increasing the polynomial reproduction: peudo-splines

B-splines subdivision schemes reproduce Π_1 only. Can we increase polynomial reproduction of B-splines?

For $m, \ell \in \mathbb{N}_0$, $\ell \leq m$, the univariate pseudo-spline (with $p \in \mathbb{R}$) is defined as the basic limit function of a subdivision scheme such that

- generates polynomials of degree m
- reproduces polynomials of degree ℓ
- has minimal support

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☞ Pseudo-splines can be derived working with their subdivision symbol and imposing the algebraic conditions for polynomial generation/reproduction

Increasing the polynomial reproduction: primal pseudo-splines

For $\ell = 0, \cdots, n-1$ primal pseudo-splines are defined by the symbols

Proposition [Daubechies et al 2003]

$$u_n^\ell(z) = 2\left(\frac{(1+z)^2}{4z}\right)^n \sum_{i=0}^{\ell} \binom{n+i-1}{i} \left(-\frac{(1-z)^2}{4z}\right)^i$$

Primal pseudo-splines were introduced to construct tight wavelet frames with high approximation order. They contain as extreme cases the schemes for uniform B-splines with odd degree and the interpolatory schemes of DD. They generates polynomials of degree $2n-1$ and reproduces polynomials of degree $2\ell+1$.

Increasing the polynomial reproduction: dual pseudo-splines

For $\ell = 0, \cdots, n-1$ dual pseudo-splines are defined by the symbols

Proposition [Dyn et al 2008]

$$u_n^\ell(z) = \frac{z+1}{z}\left(\frac{(1+z)^2}{4z}\right)^n \sum_{i=0}^{\ell} \binom{n+i-\frac{1}{2}}{i} \left(-\frac{(1-z)^2}{4z}\right)^i$$

Dual pseudo-splines contains as extreme cases the schemes for uniform B-splines with even degree and the dual DD schemes. They generates polynomials of degree $2n$ and reproduces polynomials of degree $2\ell+1$.

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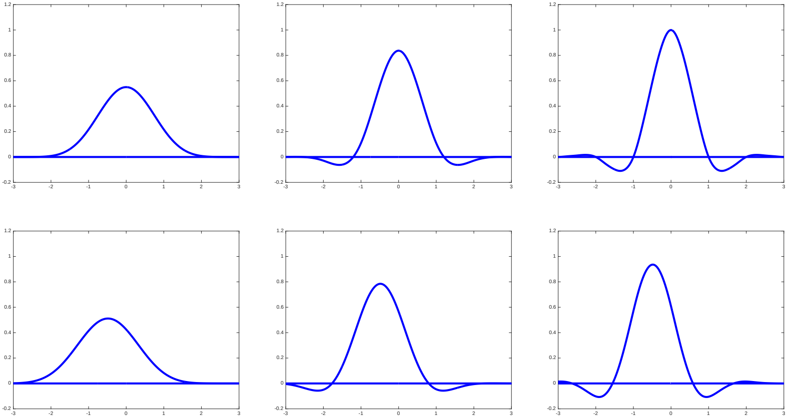
☞ $u_n^\ell(z)$ can be seen as "corrected" B-splines

Increasing the polynomial reproduction: pseudo-splines

Up: from degree five B-splines to 6-point interpolatory.

Down: From degree six B-splines to dual 6-point scheme.



Bivariate pseudo-splines

- How to construct the bivariate counterpart of pseudo-splines?
- Via a subdivision approach that is constructing the corresponding subdivision symbols...

We employ methods of *algebraic nature* to derive subdivision symbols that, besides *generating* polynomials of degree n, *reproduce* polynomials of degree up to n and have *minimal support* .

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For symmetry reasons we work with a four direction grid of the plane and with symmetric (primal) schemes. The role played by univariate B-spline is played by bivariate Box-splines.

Box-splines

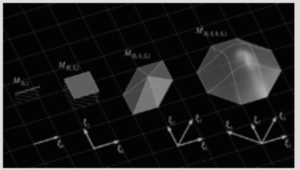
Definition [de Boor, Höllig, Riemenschneider, 1993]

Let $\boldsymbol{\Xi} = \{\boldsymbol{\xi}_1 \ldots \boldsymbol{\xi}_N\}$ a set of vectors in $\mathbb{N}^2$. For $N = 2$ the box spline is the normalized indicator function of the parallelogram formed by the two vectors (if l.i.)

$$B_{\boldsymbol{\Xi}}(\boldsymbol{x}) = \frac{1}{det(\boldsymbol{\Xi})}\chi_{\boldsymbol{\Xi}} = \begin{cases} \frac{1}{det(\boldsymbol{\Xi})}, & \boldsymbol{x} = (1-t)\boldsymbol{\xi}_1 + t\boldsymbol{\xi}_2 \\ 0, & \text{otherwise.} \end{cases}$$

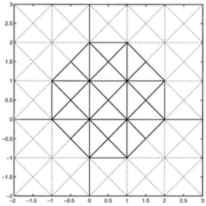
Adding a new vector ξ the box spline is recursively defined as

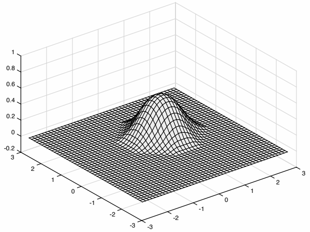
$$B_{\boldsymbol{\Xi}\cup\xi}(\boldsymbol{x}) := \int_0^1 B_{\boldsymbol{\Xi}}(\boldsymbol{x} - t\boldsymbol{\xi})\, dt = \int_0^1 B_{\boldsymbol{\Xi}}(x_1 - t\xi_1, x_2 - t\xi_2)\, dt\,.$$



Four direction Box-splines

Four direction Box-splines are based on the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$





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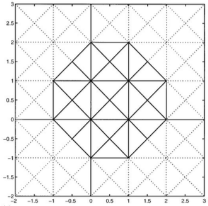

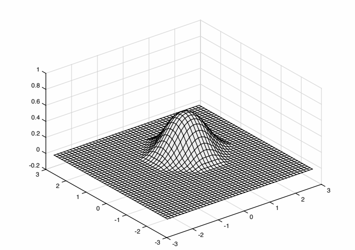

☞ Box-splines on uniform grids are refinable functions and therefore can be constructed via a (bivariate) subdivision scheme.

Bivariate subdivision schemes

Here we work with multi-index notations $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ based on which the subdivision operator is simply defined as

$$(S_{\mathbf{a}} P)_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^2} a_{\boldsymbol{\alpha} - 2\boldsymbol{\beta}} P_{\boldsymbol{\beta}}, \quad \alpha \in \mathbb{Z}^2$$

consisting of 4 different rules according to the parity of the indices.

The subdivision scheme is $\Rightarrow \begin{cases} \text{Input} \quad \mathbf{P}^{(0)} \\ \text{For} \quad k = 0, 1, \cdots \\ \qquad \mathbf{P}^{(k+1)} := S_{\mathbf{a}} \mathbf{f}^{(k)} \end{cases}$

and the subdivision symbol is the bivariate Laurent polynomial

$$a(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^2} a_{\boldsymbol{\alpha}} \, \boldsymbol{z}^{\boldsymbol{\alpha}}, \qquad \boldsymbol{z} = (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}, \quad \boldsymbol{z}^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2}.$$

Polynomial generation and reproduction for subdivision

Based on the same definition of polynomial generation and reproduction, for stable schemes (limit zero only for zero initial sequences) we can prove that

Proposition [Charina and C. 2013]

- A subdivision scheme $S_{\mathbf{a}}$ generates Π_d iff for Ξ' ={(1,-1), (-1,1) (-1,-1)}

$$a(1,1) = 4, \quad \big(D^{\boldsymbol{j}} a\big)(\epsilon) = 0 \ \text{ for } \ \epsilon \in \Xi', \quad |\boldsymbol{j}| \leq d.$$

- A subdivision scheme $S_{\mathbf{a}}$ reproduces Π_d iff for $|\boldsymbol{j}| \leq d$, $\boldsymbol{j} \in \mathbb{N}_0^2$, $\mathbf{p} \in \mathbb{R}^2$

$$\big(D^{\boldsymbol{j}} a\big)(1,1) = 4 \prod_{\ell_i=0}^{j_i-1} (p_i - \ell_i) \ \text{ and } \ \big(D^{\boldsymbol{j}} a\big)|_{\epsilon \in \Xi'}(\epsilon) = 0.$$

Construction of bivariate pseudo-splines

The simplest bivariate counterpart of pseudo-splines is based on TP

Proposition

The bivariate symbol of a TP primal univariate pseudo splines

$$\bar{a}_n^{\ell}(\boldsymbol{z}) = u_n^{\ell}(z_1)u_n^{\ell}(z_2), \quad \ell = 0, \cdots, n-1$$

generates polynomials of degree $2n-1$ and reproduces polynomials up to degree $2\ell + 1$.

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☞ However, the support of these schemes is not minimal.

Bivariate pseudo-splines

Let us consider the "extreme" element of the bivariate counterpart of primal pseudo-splines: Han and Jia in 1998 showed that there exists a unique symmetric interpolatory scheme with generation and reproduction degree $2n-1$ and minimal support (no explicit formula).

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These symbols can be represented nicely in terms of the symbols of the univariate (primal) pseudo-splines schemes.

Proposition [C., Deng and Hormann, 2015]

The minimally supported bivariate symbol

$$a_n^{n-1}(\boldsymbol{z}) = \sum_{i=1}^{n} u_i^{i-1}(z_1) u_{n-i+1}^{n-i}(z_2) - \sum_{i=1}^{n-1} u_i^{i-1}(z_1) u_{n-i}^{n-i-1}(z_2), \qquad n \geq 1$$

is interpolatory, generates and reproduces polynomials of degree $2n-1$.

The role played by Box-splines for generation/reproduction

Proposition [Charina et al. 2011]

A convergent bivariate subdivision scheme $S_{\mathbf{a}}$ generates polynomials of degree m if and only if its mask symbol can be written in the form

$$a(z) = \sum_{B_{\alpha,\beta,\gamma} \in I_m} \lambda_{\alpha,\beta,\gamma} \sigma_{\alpha,\beta,\gamma}(\boldsymbol{z}) B_{\alpha,\beta,\gamma}(\boldsymbol{z}), \quad \sum \lambda_{\alpha,\beta,\gamma} = 1$$

with $B_{\alpha,\beta,\gamma} = (\frac{1+z_1}{2})^{\alpha}(\frac{1+z_2}{2})^{\beta}(\frac{1+z_1 z_2}{2})^{\gamma}$ with the normalization $\sigma_{\alpha,\beta,\gamma}(1,1) = 1$.

It is also useful to observe that any four-directional box spline symbol is a convex combination of the shifts of some three-directional box spline symbols.

$$B_{\alpha,\beta,\gamma,\delta} = \frac{1}{2^{\delta}} \left(1 + \frac{z_1}{z_2}\right)^{\delta} B_{\alpha,\beta,\gamma}(\boldsymbol{z})$$

Bivariate pseudo-splines

These observations characterize our construction of a symmetric bivariate counterpart of primal pseudo-splines with symbols $a_n^\ell(\boldsymbol{z}),\ \ell = 0, \cdots, n-1$. We require that:

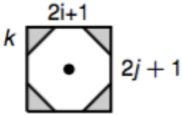
- The first element of the family $a_n^0(\boldsymbol{z})$ is a scaled four-directional box-spline of a certain degree (i.e. minimally supported for that degree)
- The last element of the family $a_n^{n-1}(\boldsymbol{z})$ is the minimally supported interpolatory scheme discovered by Han and Jia
- A generic element of the family $a_n^\ell(\boldsymbol{z}), \ell = 1, \cdots, n-2$ is symmetric (primal) and satisfy the algebraic properties for polynomial generation of degree $2n-1$ and for reproduction of degree $2\ell + 1$.

Symmetric four direction Box-splines and their symbols

The symbol of the first element of our family is the symbol of a *symmetric four-directional box-spline*

$$B_{i,j,k}(z_1,z_2) := \left(\frac{(1+z_1)^2}{4z_1}\right)^i \left(\frac{(1+z_2)^2}{4z_2}\right)^j \left(\frac{(1+z_1z_2)(z_1+z_2)}{4z_1z_2}\right)^k$$

with symmetric support given by the rectangle $[-i,i]\times[-j,j]$ with zero coefficients in the triangular regions with side length k in each corner:



- $B_{i,j,k}$ generates polynomials up to degree $2(i+j+k-\max(i,j,k))$
- $B_{i,j,k}$ reproduces polynomials of degree 1 only.

Four direction pseudo-splines with minimal support

Denoting by $a^0_{n-i}(\boldsymbol{z}) = B_{\lceil \frac{n-i}{2} \rceil, \lceil \frac{n-i}{2} \rceil, \lfloor \frac{n-i}{2} \rfloor}(z_1, z_2)$ and by $\delta(\boldsymbol{z})^{\boldsymbol{\alpha}} = \left(\frac{-(1-z_1)^2}{z_1}\right)^{\alpha_1} \left(\frac{-(1-z_2)^2}{z_2}\right)^{\alpha_2}$ we propose the family of pseudo-splines

Definition [C., Deng, Hormann, 2015]

$$a^\ell_n(\boldsymbol{z}) = \sum_{i=0}^{\ell} a^0_{n-i}(\boldsymbol{z}) \sum_{j=0}^{i} c_n^{(i,j)} \delta(\boldsymbol{z}^2)^{(i-j,j)}, \quad \ell = 0, \cdots, n-1$$

where

$$c_n^{(i,j)} = \frac{1}{4^i} \sum_{k=0}^{\lceil \frac{j}{2} \rceil} (-1)^k \binom{\left\lceil \frac{n+i-2j}{2} \right\rceil - 1 + k}{k} \binom{n-1+j-k}{j-k} \binom{n-1+i-2j}{i-j-k},$$

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☞ $$a^\ell_n(\boldsymbol{z}) = \sum_{i=0}^{\ell} a^0_{n-i}(\boldsymbol{z}) b^i_n(\boldsymbol{z}^2), \quad \ell = 1, \cdots, n-1.$$

Four direction pseudo-splines with minimal support

Proposition [C., Deng, and Hormann 2016]

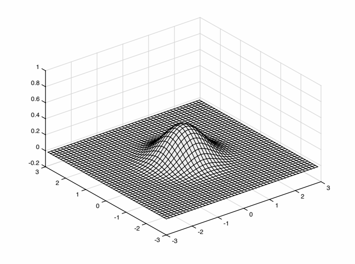
Let $a_n^\ell(\boldsymbol{z}) = \sum_{i=0}^{\ell} a_{n-i}^0(\boldsymbol{z}) b_n^i(\boldsymbol{z}^2), \quad \ell = 0, \cdots, n-1$

- $a_n^0(\boldsymbol{z})$ is the symbol of a four direction Box-spline $B_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}(\boldsymbol{z})$
- $a_n^{n-1}(\boldsymbol{z})$ is an interpolatory symbol with polynomial generation and reproduction degrees equal to $2n-1$ (H&J)
- The polynomial generation degrees of $a_n^\ell(\boldsymbol{z})$ is $2n-1$ and its polynomial reproduction is $2\ell+1$, $\ell = 0, \cdots, n-1$
- $a_n^\ell(\boldsymbol{z})$ has small support

☞ in progress: convergence, regularity, minimality of the support, ...

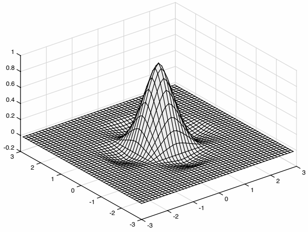
An example: $n = 2,\ \ell = 0$

$$\frac{1}{16}\begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \\ 2 & 6 & 8 & 6 & 2 \\ 1 & 4 & 6 & 4 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$



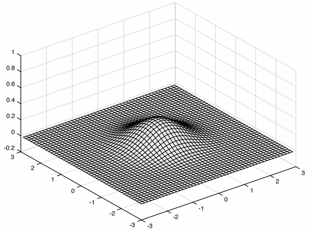
An example: $n = 2,\ \ell = 1$

$$\frac{1}{32}\begin{bmatrix} 0 & 0 & -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 10 & 18 & 10 & 0 & -1 \\ -2 & 0 & 18 & 32 & 18 & 0 & -2 \\ -1 & 0 & 10 & 18 & 10 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & -1 & 0 & 0 \end{bmatrix}$$



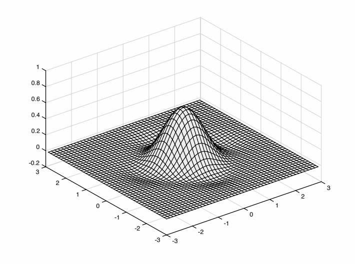
An example: $n = 3, \ \ell = 0$

$$\frac{1}{256}\begin{bmatrix} 0 & 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 8 & 23 & 32 & 23 & 8 & 1 \\ 4 & 23 & 56 & 74 & 56 & 23 & 4 \\ 6 & 32 & 74 & 96 & 74 & 32 & 6 \\ 4 & 23 & 56 & 74 & 56 & 23 & 4 \\ 1 & 8 & 23 & 32 & 23 & 8 & 1 \\ 0 & 1 & 4 & 6 & 4 & 1 & 0 \end{bmatrix}$$



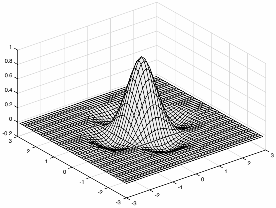
An example: $n = 3,\ \ell = 1$

$$
\frac{1}{256}
\begin{bmatrix}
0 & 0 & 0 & -3 & -6 & -3 & 0 & 0 & 0 \\
0 & 0 & -2 & -8 & -12 & -8 & -2 & 0 & 0 \\
0 & -2 & -4 & 14 & 32 & 14 & -4 & 2 & 0 \\
-3 & -8 & 14 & 80 & 122 & 80 & 14 & -8 & -3 \\
-6 & -12 & 32 & 122 & 168 & 122 & 32 & & \\
-3 & -8 & 14 & 80 & 122 & 80 & 14 & & \\
0 & -2 & -4 & 14 & 32 & 14 & -4 & & \\
0 & 0 & -2 & -8 & -12 & -8 & -2 & & \\
0 & 0 & 0 & -3 & -6 & -3 & 0 & &
\end{bmatrix}
$$



An example: $n = 3,\ \ell = 2$

$$\frac{1}{2048}\begin{bmatrix} 0 & 0 & 0 & 0 & 12 & 24 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & -108 & -200 & -108 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & -108 & 0 & 696 & 1200 & 696 & 0 & -108 & 0 & 12 \\ 24 & 0 & -200 & 0 & 1200 & 2048 & 1200 & 0 & -200 & 0 & 2 \\ 12 & 0 & -108 & 0 & 696 & 1200 & 696 & 0 & -108 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 8 & 0 & -108 & -200 & -108 & 0 & 8 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 12 & 24 & 12 & 0 & 0 & 0 & \end{bmatrix}$$



Thank you for your attention!

From Box-splines to four-directional pseudo-splines

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Joint work with: Chongyang Deng and Kai Hormann

IM-Workshop on "Signals, Images, and Approximation", Bernried, February 29-March 4