Limit stencils of non-stationary approximating schemes and their applications

Paola Novara

Department of Science and High Technology, University of Insubria, Italy

Joint work with:

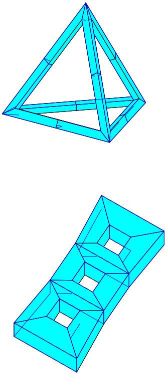
Lucia Romani, University of Milano-Bicocca, Italy

Workshop on "Applied Approximation, Signals and Images"

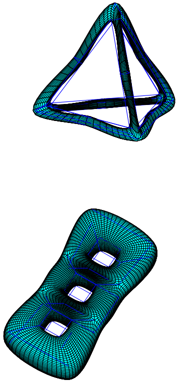
Bernierd, February 29 - March 4, 2016

Motivations

Goal: Construction of interpolating surfaces of *good quality* from meshes with arbitrary manifold topology



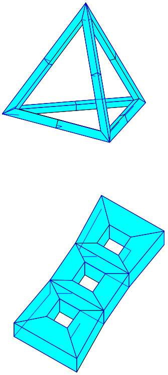
Initial meshes



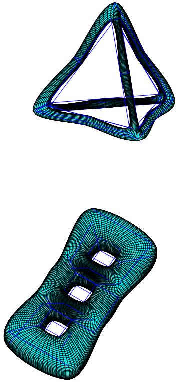
Interpolatory scheme

Motivations

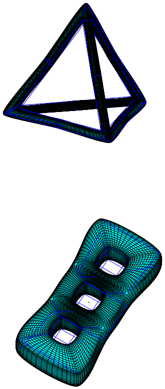
Goal: Construction of interpolating surfaces of *good quality* from meshes with arbitrary manifold topology



Initial meshes



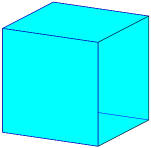
Interpolatory scheme

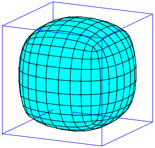


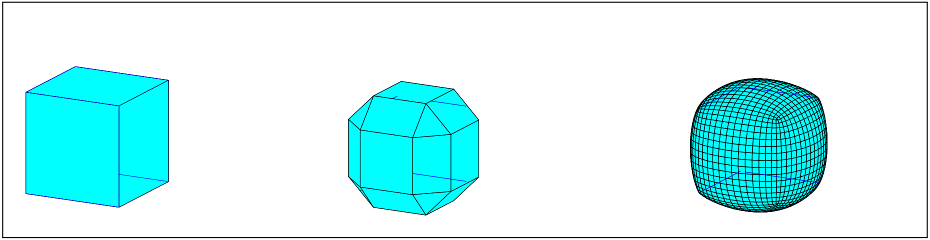
New method

Motivations

How? Using approximating subdivision schemes with a *preprocessing step* on the control mesh by means of the limit stencil of the scheme.

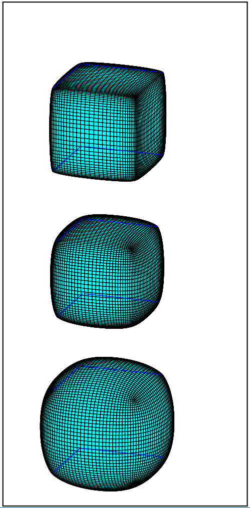






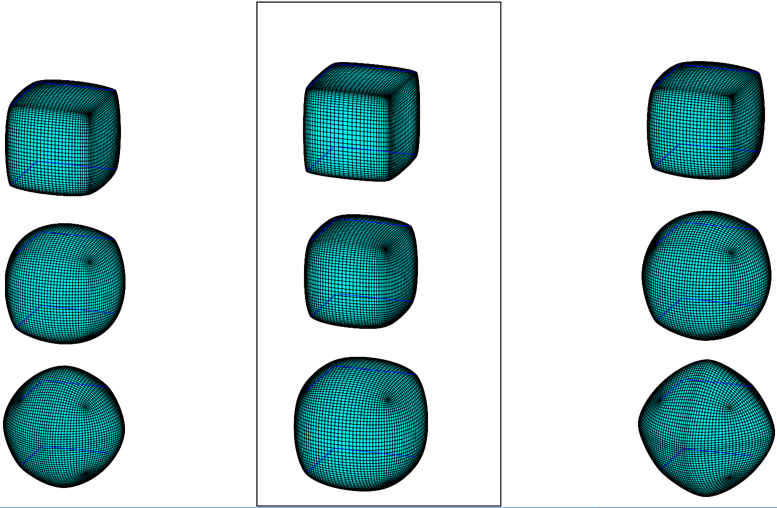
Motivations

And to gain flexibility? The use of the preprocessing step together with *non-stationary subdivision rules* let us gain two shape parameters.



Motivations

And to gain flexibility? The use of the preprocessing step together with *non-stationary subdivision rules* let us gain two shape parameters.



The general algorithm

Input: Initial control points $\mathbf{P}^{(0)}$

- Apply once the subdivision rules to $\mathbf{P}^{(0)}$ to compute the points $\tilde{\mathbf{P}}^{(0)}$;

The general algorithm

Input: Initial control points $\mathbf{P}^{(0)}$

- Apply once the subdivision rules to $\mathbf{P}^{(0)}$ to compute the points $\tilde{\mathbf{P}}^{(0)}$;
- Compute the limit positions $\tilde{\mathcal{L}}$

The general algorithm

Input: Initial control points $\mathbf{P}^{(0)}$

- Apply once the subdivision rules to $\mathbf{P}^{(0)}$ to compute the points $\tilde{\mathbf{P}}^{(0)}$;
- Compute the limit positions $\tilde{\mathcal{L}} \longrightarrow$ We need the limit stencil!

The general algorithm

Input: Initial control points $\mathbf{P}^{(0)}$

- Apply once the subdivision rules to $\mathbf{P}^{(0)}$ to compute the points $\tilde{\mathbf{P}}^{(0)}$;
- Compute the limit positions $\tilde{\mathcal{L}} \longrightarrow$ We need the limit stencil!
- Compute the new control points $\mathbf{Q}^{(0)}$ as

$$Q_i^{(0)} = P_i^{(0)} + \alpha(\tilde{P}_i^{(0)} - \tilde{\mathcal{L}}_i), \qquad \alpha \in \mathbb{R};$$

The general algorithm

Input: Initial control points $\mathbf{P}^{(0)}$

- Apply once the subdivision rules to $\mathbf{P}^{(0)}$ to compute the points $\tilde{\mathbf{P}}^{(0)}$;
- Compute the limit positions $\tilde{\mathcal{L}} \longrightarrow$ We need the limit stencil!
- Compute the new control points $\mathbf{Q}^{(0)}$ as

$$Q_i^{(0)} = P_i^{(0)} + \alpha(\tilde{P}_i^{(0)} - \tilde{\mathcal{L}}_i), \qquad \alpha \in \mathbb{R};$$

- Apply the subdivision scheme to the control points $\mathbf{Q}^{(0)}$.

The general algorithm

Input: Initial control points $\mathbf{P}^{(0)}$

- Apply once the subdivision rules to $\mathbf{P}^{(0)}$ to compute the points $\tilde{\mathbf{P}}^{(0)}$;
- Compute the limit positions $\tilde{\mathcal{L}} \longrightarrow$ We need the limit stencil!
- Compute the new control points $\mathbf{Q}^{(0)}$ as

$$Q_i^{(0)} = P_i^{(0)} + \alpha(\tilde{P}_i^{(0)} - \tilde{\mathcal{L}}_i), \qquad \alpha \in \mathbb{R};$$

- Apply the subdivision scheme to the control points $\mathbf{Q}^{(0)}$.

Output: A limit curve/surface interpolating the initial control points $\mathbf{P}^{(0)}$.

The general algorithm

Input: Initial control points $\mathbf{P}^{(0)}$

- Apply once the subdivision rules to $\mathbf{P}^{(0)}$ to compute the points $\tilde{\mathbf{P}}^{(0)}$;
- Compute the limit positions $\tilde{\mathcal{L}} \longrightarrow$ We need the limit stencil!
- Compute the new control points $\mathbf{Q}^{(0)}$ as

$$Q_i^{(0)} = P_i^{(0)} + \alpha(\tilde{P}_i^{(0)} - \tilde{\mathcal{L}}_i), \qquad \alpha \in \mathbb{R};$$

- Apply the subdivision scheme to the control points $\mathbf{Q}^{(0)}$.

Output: A limit curve/surface interpolating the initial control points $\mathbf{P}^{(0)}$.

Limit stencils of stationary schemes

The subdivision rules could be written in a matrix form, where the subdivision matrix S is the same at each subdivision level

$$\mathbf{P}^{(k+1)} = S\mathbf{P}^{(k)} = S^{k+1}\mathbf{P}^{(0)}.$$

- Eigen-decomposition of S

$$\mathbf{P}^{(k+1)} = S^{k+1}\mathbf{P}^{(0)} = VD^{k+1}W\mathbf{P}^{(0)}, \text{ where } D^{k+1} = \begin{pmatrix} \lambda_0^{k+1} & 0 & \dots & 0 \\ 0 & \lambda_1^{k+1} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_{n-1}^{k+1} \end{pmatrix}.$$

- For the convergence of the scheme $1 = \lambda_0 < \lambda_i, \forall i = 1, \dots, n-1$ and $\mathbf{v}_0 = \mathbf{1}$

$$\lim_{k\to+\infty} \mathbf{P}^{(k+1)} = V\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix} W\mathbf{P}^{(0)} = \mathbf{v}_0\tilde{\mathbf{w}}_0^T\mathbf{P}^{(0)} = \begin{pmatrix} \tilde{\mathbf{w}}_0^T \\ \tilde{\mathbf{w}}_0^T \\ \vdots \\ \tilde{\mathbf{w}}_0^T \end{pmatrix} \mathbf{P}^{(0)}.$$

Limit stencils of non-stationary schemes

The subdivision rules could be written in a matrix form, where the subdivision matrix S_k depends on the subdivision level

$$\mathbf{P}^{(k+1)} = S_k \mathbf{P}^{(k)} = S_k \cdot S_{k-1} \cdot \ldots \cdot S_0 \mathbf{P}^{(0)}.$$

The limit stencil has to be derived from the subdivision process.

How?

- geometrical point of view: study the evolution of the position of the vertices;
- algebraic point of view: study the behavior of the subdivision matrices at different subdivision levels.

Limit stencils of non-stationary schemes

The subdivision rules could be written in a matrix form, where the subdivision matrix S_k depends on the subdivision level

$$\mathbf{P}^{(k+1)} = S_k \mathbf{P}^{(k)} = S_k \cdot S_{k-1} \cdot \ldots \cdot S_0 \mathbf{P}^{(0)}.$$

The limit stencil has to be derived from the subdivision process.

How?

- geometrical point of view: study the evolution of the position of the vertices;
- algebraic point of view: study the behavior of the subdivision matrices at different subdivision levels.

Outline

We illustrate our strategy to compute the limit stencil of

- primal/dual univariate non-stationary subdivision schemes,
- primal/dual bivariate non-stationary subdivision schemes.

We test the method on some examples

- a non-stationary version of Chaikin's scheme,
- two non-stationary versions of cubic B-splines.

Outline

We illustrate our strategy to compute the limit stencil of

- primal/dual univariate non-stationary subdivision schemes,
- primal/dual bivariate non-stationary subdivision schemes.

We test the method on some examples

- a non-stationary version of Chaikin's scheme,
- two non-stationary versions of cubic B-splines.

- a non-stationary version of Doo-Sabin's scheme,
- the non-stationary version of Catmull-Clark's scheme proposed by Fang, W. Ma, G. Wang, (2014) $\longrightarrow$ work in progress!

Outline

We illustrate our strategy to compute the limit stencil of

- primal/dual univariate non-stationary subdivision schemes,
- primal/dual bivariate non-stationary subdivision schemes.

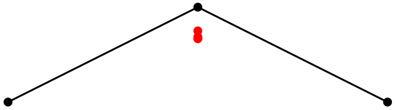
We test the method on some examples

- a non-stationary version of Chaikin's scheme,
- two non-stationary versions of cubic B-splines.

- a non-stationary version of Doo-Sabin's scheme,
- the non-stationary version of Catmull-Clark's scheme proposed by Fang, W. Ma, G. Wang, (2014) $\longrightarrow$ work in progress!

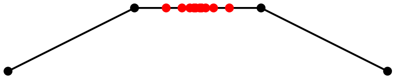
Limit stencil of primal/dual univariate subdivision schemes

- Primal schemes: we study the evolution of the central point.



[Cubic B-splines]

- Dual schemes: we study the evolution of the points on the central edge.



[Chaikin's scheme]

Limit stencil of univariate subdivision schemes

- From the subdivision rules compute the subdivision matrix S_ℓ.
- To find the limit stencil we study $\lim_{k\to+\infty}\prod_{\ell=0}^{k} S_\ell$
- Eigen-decomposition of S_ℓ

$$\lim_{k\to+\infty}\prod_{\ell=0}^{k} S_\ell = \lim_{k\to+\infty}\prod_{\ell=0}^{k} V_\ell D_\ell W_\ell$$

- Expand $\prod_{\ell=0}^{k} V_\ell D_\ell W_\ell$ as

$$\prod_{\ell=0}^{k} V_\ell D_\ell W_\ell = V_k \underbrace{(D_k W_k V_{k-1} \dots V_0 D_0)}_{T_k} W_0.$$

- Compute $\lim_{k\to+\infty} V_k T_k W_0$.

A non-stationary version of Chaikin's scheme

Subdivision rules [M. Fang, W. Ma, G. Wang, 2010]

$$\begin{aligned} P_{2i}^{(k+1)} &= w_k P_{i-1}^{(k)} + (1-w_k) P_i^{(k)}, \\ P_{2i+1}^{(k+1)} &= (1-w_k) P_i^{(k)} + w_k P_{i+1}^{(k)} \end{aligned}$$

with $w_k = \frac{1}{2(1+v_k)}$, $v_k = \frac{1}{2}\left(e^{\mathrm{i}\frac{\lambda}{2^{k+1}}} + e^{-\mathrm{i}\frac{\lambda}{2^{k+1}}}\right)$, $\lambda \in [0,\pi) \cup \mathrm{i}\mathbb{R}^+$.

Subdivision matrix:

$$S_k = \begin{pmatrix} 1-w_k & w_k \\ w_k & 1-w_k \end{pmatrix}$$

Limit Stencil: $\left[\frac{1}{2}, \frac{1}{2}\right]$

Two non-stationary versions of cubic B-splines

Subdivision rules [Romani et al. 2016]

$$
\begin{aligned}
P_{2i}^{(k)} &= \frac{\alpha_k}{8} P_{i-1}^{(k)} + \left(1 - \frac{\alpha_k}{4}\right) P_i^{(k)} + \frac{\alpha_k}{8} P_{i+1}^{(k)},\\
P_{2i+1}^{(k)} &= \frac{1}{2} P_i^{(k)} + \frac{1}{2} P_{i+1}^{(k)}.
\end{aligned}
$$

Subdivision matrix:

$$
S_k = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{\alpha_k}{8} & 1 - \frac{\alpha_k}{4} & \frac{\alpha_k}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}
$$

Limit Stencil: $\left[\frac{1-\gamma}{2}, \gamma, \frac{1-\gamma}{2}\right]$

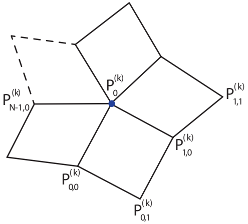
- if $\alpha_k = \frac{2}{1+\cos\left(\frac{\lambda}{2^{k+1}}\right)}$, $\lambda \in [0, \pi) \cup \mathrm{i}\mathbb{R}^+$, $\gamma = \cot\left(\frac{\lambda}{2}\right)\left(\frac{1}{\lambda} - \cot\lambda\right)$;
- if $\alpha_k = \frac{k+2(1-2^{(\lambda+1)})}{2^\lambda(k+1)}$, $\lambda \in \mathbb{R}^+$, $\gamma = \frac{1}{2(1-f(\lambda))^{g(\lambda)}}$ where $f(\lambda) = \frac{1}{2} - 2^{-(\lambda+2)}$ and $g(\lambda) = 1 + \frac{2^{-\lambda}-1}{2^{\lambda+1}-1}$.

Limit stencil of primal bivariate subdivision schemes

$$\begin{pmatrix} P_0^{(k+1)} \\ \mathbf{P}_0^{(k+1)} \\ \mathbf{P}_1^{(k+1)} \\ \vdots \\ \mathbf{P}_{N-1}^{(k+1)} \end{pmatrix} = \underbrace{\left(\begin{array}{c|cccc} a & \mathbf{r}^T & \mathbf{r}^T & \cdots & \mathbf{r}^T \\ \hline \mathbf{c} & M_{0,k} & M_{1,k} & \cdots & M_{N-1,k} \\ \mathbf{c} & M_{N-1,k} & M_{0,k} & \cdots & M_{N-2,k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{c} & M_{1,k} & M_{2,k} & \cdots & M_{0,k} \end{array}\right)}_{S_k^{[N]}} \begin{pmatrix} P_0^{(k)} \\ \mathbf{P}_0^{(k)} \\ \mathbf{P}_1^{(k)} \\ \vdots \\ \mathbf{P}_{N-1}^{(k)} \end{pmatrix}$$

with $a \in \mathbb{R}$, $\mathbf{r}, \mathbf{c} \in \mathbb{R}^{p\times 1}$, $M_{i,k} \in \mathbb{R}^{p\times p}$, $P_0^{(k)} \in \mathbb{R}$, $\mathbf{P}_i^{(k)} \in \mathbb{R}^{p\times 1}$.

- The k-level subdivision matrix $S_k^{[N]}$ is an hybrid block-circulant matrix.
- Dimension of $S_k^{[N]} \in \mathbb{R}^{(pN+1)\times(pN+1)}$ increases with N, the eigen-decomposition becomes computationally difficult.



Limit stencil of primal bivariate subdivision schemes

- We transform $S_k^{[N]}$ in a block-circulant matrix

$$R_k^{[N]} = \begin{pmatrix} R_{0,k} & R_{1,k} & \cdots & R_{N-1,k} \\ R_{N-1,k} & R_{0,k} & \cdots & R_{N-2,k} \\ \vdots & \ddots & \ddots & \vdots \\ R_{1,k} & R_{2,k} & \cdots & R_{0,k} \end{pmatrix}, \text{ with } R_{i,k} = \begin{pmatrix} \frac{a}{N} & \mathbf{r}^T \\ \frac{\mathbf{c}}{N} & M_{i,k} \end{pmatrix}$$

- We apply the discrete Fourier transform to obtain

$$\hat{S}_k^{[N]} = \begin{pmatrix} \hat{S}_{0,k} & 0 & \cdots & 0 \\ 0 & \hat{S}_{1,k} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{S}_{N-1,k} \end{pmatrix}, \quad \begin{array}{l} \hat{S}_{\nu,k} = \sum_{j=0}^{N-1} R_{j,k}\omega^{j\nu}, \\ \text{with } \nu = 0,\dots,N-1,\ \omega = e^{\frac{2\pi i}{N}}. \end{array}$$

- We focus on $\hat{S}_{0,k}$ since it contains the dominant eigenvalue λ_0 and the eigenvector $\mathbf{v}_0 = \mathbf{1}$

$$\hat{S}_{0,k} = \sum_{i=0}^{N-1} R_{i,k} = \begin{pmatrix} a & N\mathbf{r}^T \\ \mathbf{c} & \sum_{i=0}^{N-1} M_{i,k} \end{pmatrix} \in \mathbb{R}^{(p+1)\times(p+1)}$$

Limit stencil of primal bivariate subdivision scheme

- We study $\lim_{k\to+\infty}\prod_{\ell=0}^{k}\hat{S}_{0,\ell}$ using the eigen-decomposition of $\hat{S}_{0,\ell}$

$$\lim_{k\to+\infty}\prod_{\ell=0}^{k}\hat{S}_{0,\ell}=\lim_{k\to+\infty}\prod_{\ell=0}^{k}V_\ell D_\ell W_\ell.$$

- We expand $\prod_{\ell=0}^{k}V_\ell D_\ell W_\ell$ as

$$\prod_{\ell=0}^{k}V_\ell D_\ell W_\ell = V_k\underbrace{\left(D_k W_k V_{k-1}\dots V_0 D_0\right)}_{T_k}W_0.$$

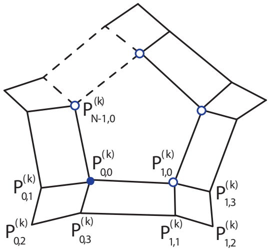
- We compute $Z=\lim_{k\to+\infty}V_k T_k W_0$.
- $Z=\begin{pmatrix} z_{1,1} & N\tilde{\mathbf{z}}_1^T \\ \tilde{\mathbf{c}} & \tilde{M}\end{pmatrix}$ preserves the structure of $\hat{S}_{0,k}$.
- The limit stencil is $[z_{1,1},\underbrace{\tilde{\mathbf{z}}_1^T,\dots,\tilde{\mathbf{z}}_1^T}_{N\text{ times}}]$.

Limit stencil of dual bivariate subdivision schemes

$$\begin{pmatrix} \mathbf{P}_0^{(k+1)} \\ \mathbf{P}_1^{(k+1)} \\ \vdots \\ \mathbf{P}_{N-1}^{(k+1)} \end{pmatrix} = \underbrace{\begin{pmatrix} M_{0,k} & M_{1,k} & \cdots & M_{N-1,k} \\ M_{N-1,k} & M_{0,k} & \cdots & M_{N-2,k} \\ \vdots & \ddots & \ddots & \vdots \\ M_{1,k} & M_{2,k} & \cdots & M_{0,k} \end{pmatrix}}_{S_k^{[N]}} \begin{pmatrix} \mathbf{P}_0^{(k)} \\ \mathbf{P}_1^{(k)} \\ \vdots \\ \mathbf{P}_{N-1}^{(k)} \end{pmatrix}$$

with $M_{i,k} \in \mathbb{R}^{p \times p}, \mathbf{P}_i^{(k)} \in \mathbb{R}^{p \times 1}$.

- The k-level subdivision matrix is a block-circulant matrix.
- Dimension of $S_k^{[N]} \in \mathbb{R}^{pN \times pN}$ increases with N, the eigen-decomposition becomes computationally difficult.



Limit stencil of dual bivariate subdivision schemes

- We apply the discrete Fourier transform to obtain

$$\hat{S}_k^{[N]} = \begin{pmatrix} \hat{S}_{0,k} & 0 & \cdots & 0 \\ 0 & \hat{S}_{1,k} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{S}_{N-1,k} \end{pmatrix}, \quad \hat{S}_{\nu,k} = \sum_{j=0}^{N-1} M_{j,k}\omega^{j\nu}, \text{ with } \nu = 0,\dots,N-1,\ \omega = e^{\frac{2\pi \mathrm{i}}{N}}.$$

- We focus on $\hat{S}_{0,k}$ since it contains the dominant eigenvalue λ_0 and the eigenvector $\mathbf{v}_0 = \mathbf{1}$

$$\hat{S}_{0,k} = \sum_{i=0}^{N-1} M_{i,k} \in \mathbb{R}^{p\times p}.$$

- We study $\lim_{k\to+\infty} \prod_{\ell=0}^{k} \hat{S}_{0,\ell}$ using the eigen-decomposition of $\hat{S}_{0,\ell}$

$$\lim_{k\to+\infty} \prod_{\ell=0}^{k} \hat{S}_{0,\ell} = \lim_{k\to+\infty} \prod_{\ell=0}^{k} V_\ell D_\ell W_\ell.$$

Limit stencil of dual non-stationary schemes

- We expand $\prod_{\ell=0}^{k} V_\ell D_\ell W_\ell$ as

$$\prod_{\ell=0}^{k} V_\ell D_\ell W_\ell = V_k \underbrace{\left(D_k W_k V_{k-1} \dots V_0 D_0\right)}_{T_k} W_0.$$

- We compute $Z = \lim_{k\to+\infty} V_k T_k W_0 = \begin{pmatrix} N\mathbf{z}_1^T \\ \vdots \\ N\mathbf{z}_p^T \end{pmatrix}$

- The limit stencil is $\underbrace{[\mathbf{z}_1^T, \dots, \mathbf{z}_1^T]}_{N \text{ times}}$.

A non-stationary version of Doo-Sabin's scheme

Subdivision rules [Fang, W. Ma, G. Wang, 2014]

$P_\ell^{(k+1)} = \alpha_N^{(k)} P_\ell^{(k)} + \beta_N^{(k)} (P_{\ell-1}^{(k)} + P_{\ell+1}^{(k)}) + \gamma_N^{(k)} \sum_{j=1, j \neq \{\ell-1,\ell,\ell+1\}}^{N} P_j^{(k)}$,
$\ell = 1, \ldots, N$,

where $\alpha_N^{(k)} = \frac{1+Nv_k(1+v_k)}{N(1+v_k)^2}$, $\beta_N^{(k)} = \frac{Nv_k+2}{2N(1+v_k)^2}$, $\gamma_N^{(k)} = \frac{1}{N(1+v_k)^2}$, and
$v_k = \frac{1}{2}\left(e^{\mathrm{i}\frac{\lambda}{2^{k+1}}} + e^{-\mathrm{i}\frac{\lambda}{2^{k+1}}}\right)$, $\lambda \in [0,\pi) \cup \mathrm{i}\mathbb{R}^+$.

First transformed block of the subdivision matrix:

$$\hat{S}_{0,k} = 1$$

Limit Stencil: $\left[\frac{1}{N}, \ldots, \frac{1}{N}\right]$

The general algorithm

Input: Initial control points $\mathbf{P}^{(0)}$

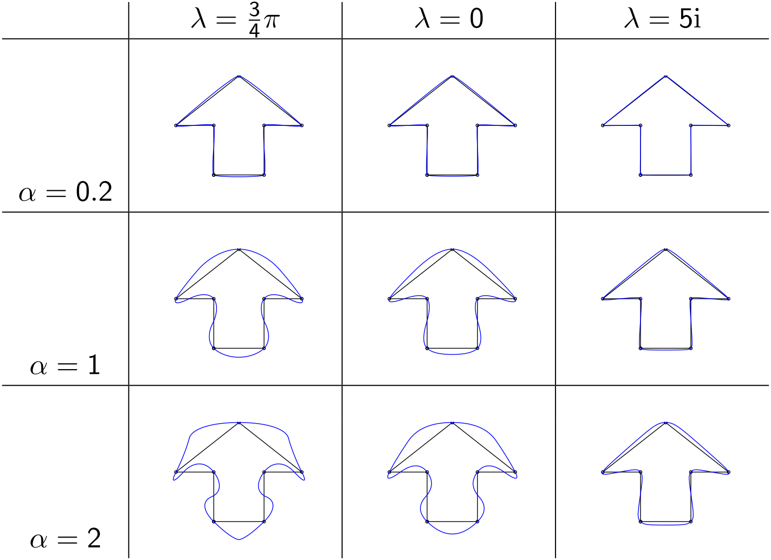
- Apply once the subdivision rules to $\mathbf{P}^{(0)}$ to compute the points $\tilde{\mathbf{P}}^{(0)}$;
- Compute the limit positions $\tilde{\mathcal{L}}$;
- Compute the new control points $\mathbf{Q}^{(0)}$ as

$$Q_i^{(0)} = P_i^{(0)} + \alpha(\tilde{P}_i^{(0)} - \tilde{\mathcal{L}}_i), \qquad \alpha \in \mathbb{R};$$

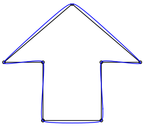
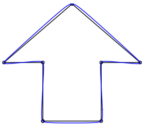
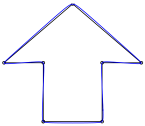
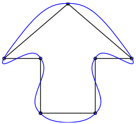
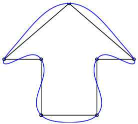
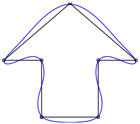
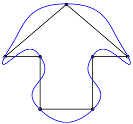
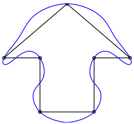
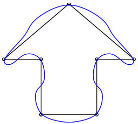
- Apply the subdivision scheme to the control points $\mathbf{Q}^{(0)}$.

Output: A limit curve/surface interpolating the initial control points $\mathbf{P}^{(0)}$.

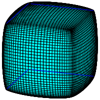
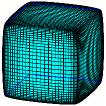
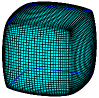
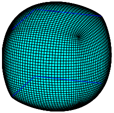
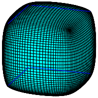
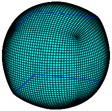
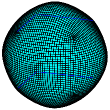
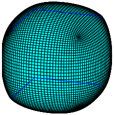
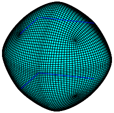
A non-stationary version of Chaikin's scheme



A non-stationary version of cubic B-spline

	$\lambda = \frac{5}{6}\pi$	$\lambda = 0$	$\lambda = 3i$
$\alpha = 0.2$			
$\alpha = 0.8$			
$\alpha = 1.2$			

A non-stationary version of Doo-Sabin's scheme

	$\lambda = \frac{5}{6}\pi$	$\lambda = 0$	$\lambda = 3\mathrm{i}$
$\alpha = 0.5$			
$\alpha = 1$			
$\alpha = 1.5$			

References

- D. Doo, M. Sabin, Behavior of recursive division surfaces near extraordinary points, Comput. Aided Design 10(6) (1978), 356-360.
- M. Fang, W. Ma, G. Wang, A generalized curve subdivision scheme of arbitrary order with a tension parameter, Comput. Aided Geom.Design 27(9) (2010), 720-733.
- M. Fang, W. Ma, G. Wang, A generalized surface subdivision scheme of arbitrary order with a tension parameter, Comput. Aided Design 49 (2014), 8–17.
- P. Novara, Interpolation of polylines by non-stationary Chaikin's scheme, submitted to MASCOT2015 Proceedings
- L. Romani, V. Hernandez-Mederos, J.Estrada-Sarlabous, Exact evaluation of a class of non-stationary approximating subdivision algorithms and related applications, IMA J. Numerical Analysis 36(1) (2016), 380–399.

Thank you for your attention!