Hermite Subdivision on Manifolds

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Overview

- What is subdivison?
  - Curve subdivision
  - Surface subdivision
- Mathematical description
- Subdivision on manifolds
- C^1 convergence analysis
- Results for Hermite schemes
- Current research

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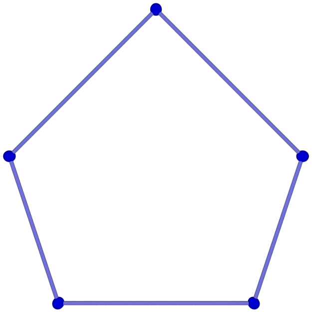

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  - successive refinement of control polygon by some rule (fixed ratio)
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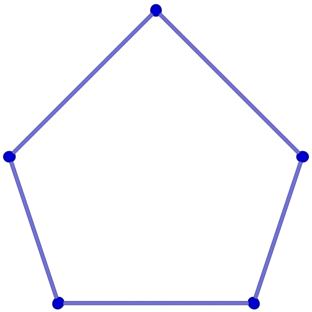

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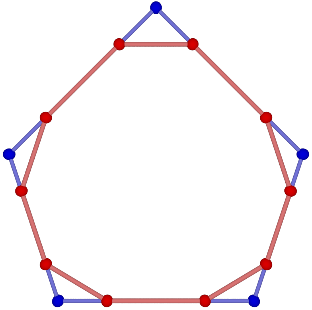

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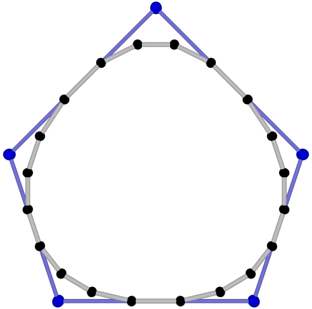

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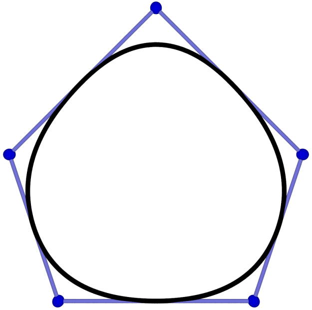

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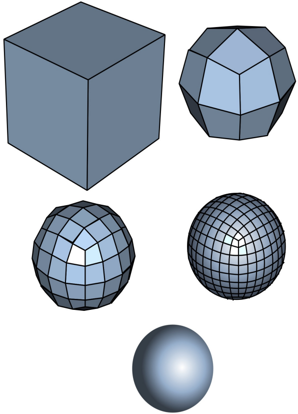

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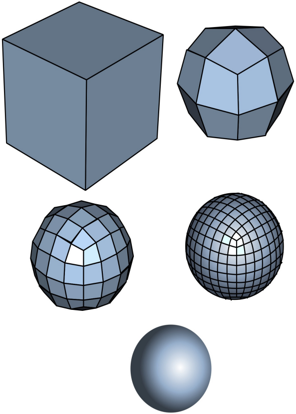

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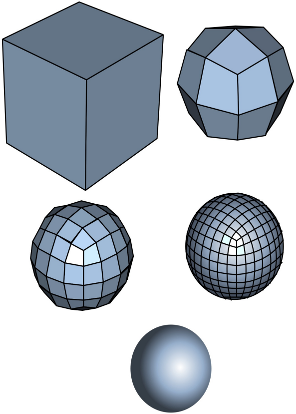

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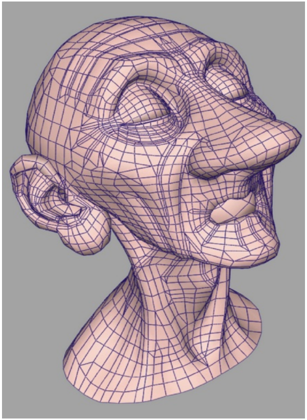

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Mathematical description

Required properties for subdivision:

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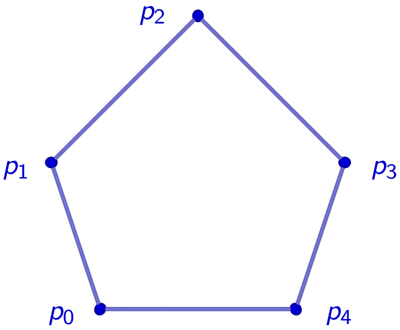

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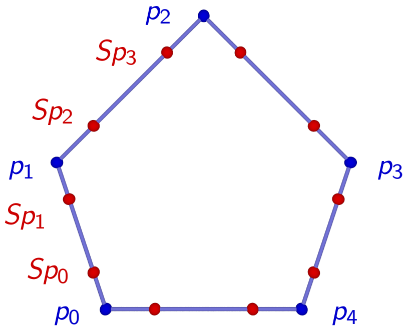

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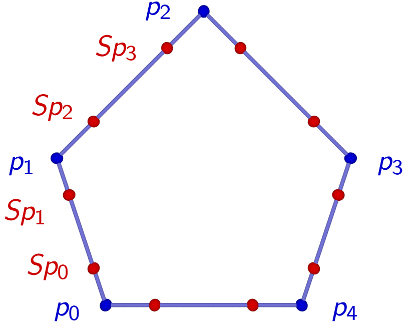

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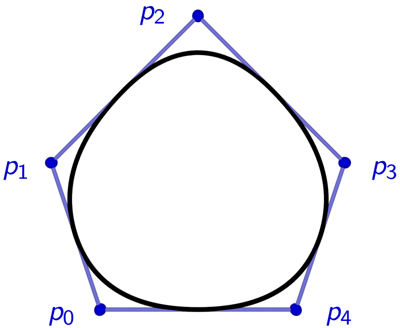
- shift invariance: $SL = L^2 S$
- local definition
- achieve some degree of smoothness, i.e. C^1 or C^2



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$$(Sp)_{2i} = \sum_{j \in \mathbb{Z}} a_{-2j} p_{i+j} \quad \text{and} \quad (Sp)_{2i+1} = \sum_{j \in \mathbb{Z}} a_{-2j+1} p_{i+j}.$$

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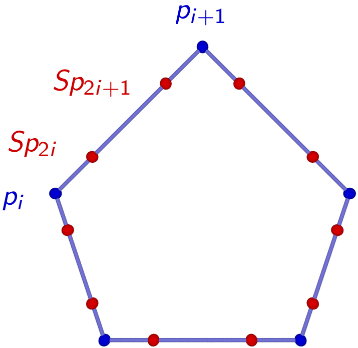

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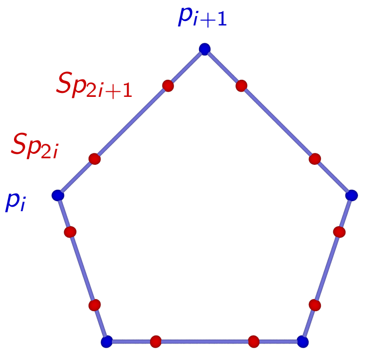

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- Restrictions to the mask:
  - $a_i \neq 0$ for only finitely many i (locality)
  - $\sum_{j\in\mathbb{Z}} a_{i-2j} = 1$ for all i (translation invariance)

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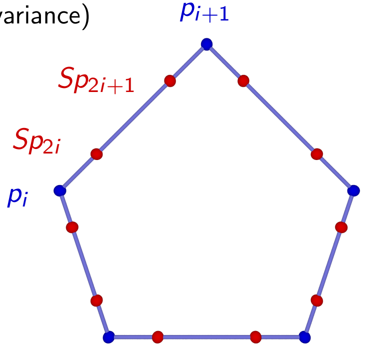

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- Parametrise polygons $S^n p$ by piecewise linear functions $\mathcal{F}_n : \mathbb{R} \to V$ such that

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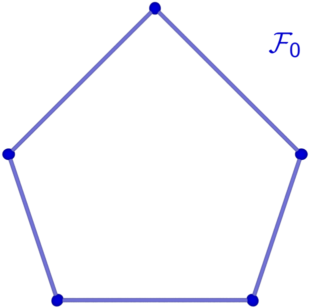

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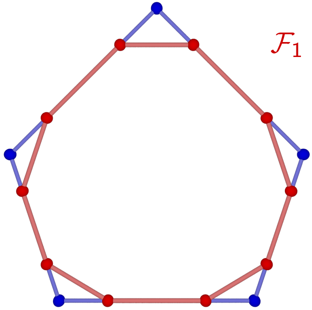

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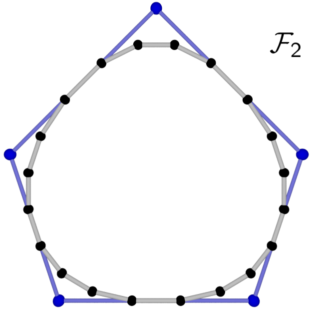

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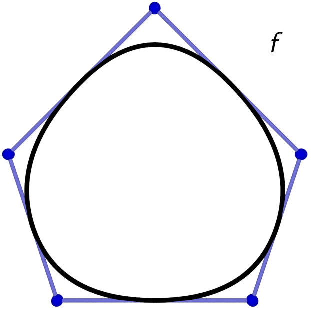

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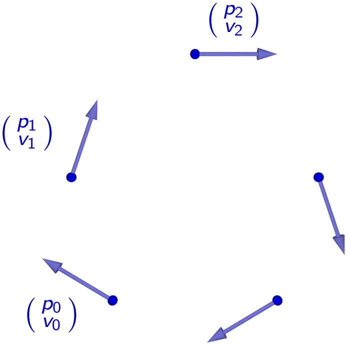
$$S\binom{p}{v}_i = \sum_{j\in\mathbb{Z}} \begin{pmatrix} a_{i-2j} & b_{i-2j} \\ c_{i-2j} & d_{i-2j} \end{pmatrix} \binom{p_j}{v_j}.$$



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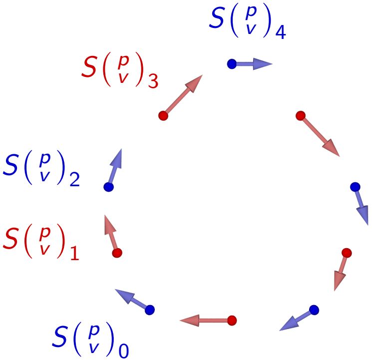

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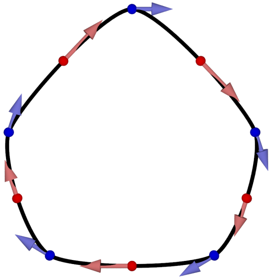
- C^1 convergence: there exists $f \in C^1(\mathbb{R}, V)$ such that

$$\mathcal{F}_n\left(\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} S^n \binom{p}{v}\right) \to \binom{f}{f'}$$

as $n \to \infty$.



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- In special geometries strong results on convergence are possible, e.g. Ebner (2014).

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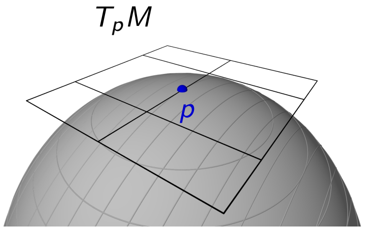
4. $(Sp)_i = \mathbb{E}X$, where $\mathbb{P}\{X = p_j\} = a_{i-2j}$

Subdivision on manifolds

M	manifold
T_pM	tangent space at p



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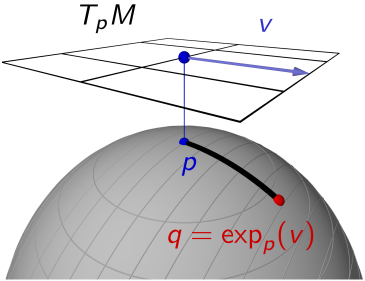

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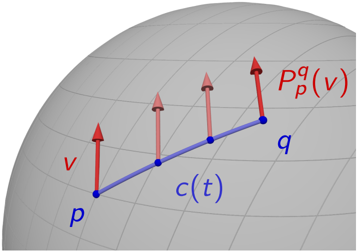

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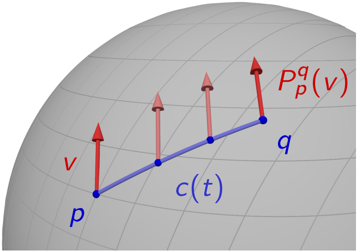

Example: $M = \mathrm{SO}(3)$ (rotations), $T_{\mathbb{I}}M = \mathfrak{so}(3)$ (skew-symmetric matrices)

$$\exp_{\mathbb{I}}(v) = \sum_{k=0}^{\infty} \frac{1}{k!} v^k \quad \text{and} \quad P_{\mathbb{I}}^q(v) = c(\tfrac{1}{2})\, v\, c(\tfrac{1}{2}),$$

where c is the "geodesic" connecting $\mathbb{I}$ and q.

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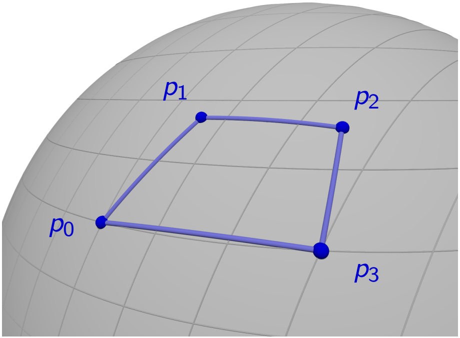

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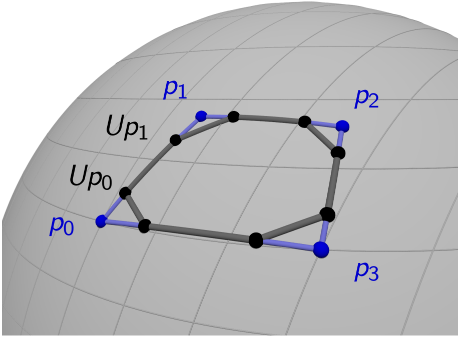

Subdivision on manifolds

Define operations:

$$p + v \quad \rightsquigarrow \quad p \oplus v := \exp_p(v)$$
$$q - p \quad \rightsquigarrow \quad q \ominus p = \exp_p^{-1}(q)$$

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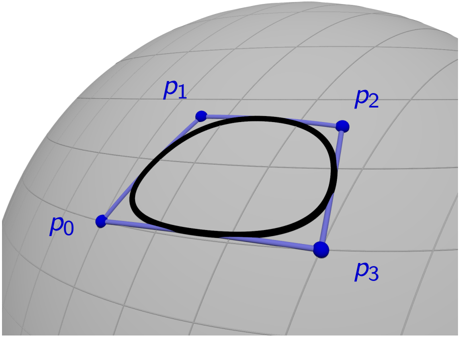

Subdivision on manifolds

Hermite case

$$S\binom{p}{v}_i = \sum_{j\in\mathbb{Z}} \begin{pmatrix} a_{i-2j} & b_{i-2j} \\ c_{i-2j} & d_{i-2j} \end{pmatrix} \begin{pmatrix} p_j \\ v_j \end{pmatrix} = \begin{pmatrix} \sum_{j\in\mathbb{Z}} a_{i-2j} p_j + b_{i-2j} v_j \\ \sum_{j\in\mathbb{Z}} c_{i-2j} p_j + d_{i-2j} v_j \end{pmatrix}$$

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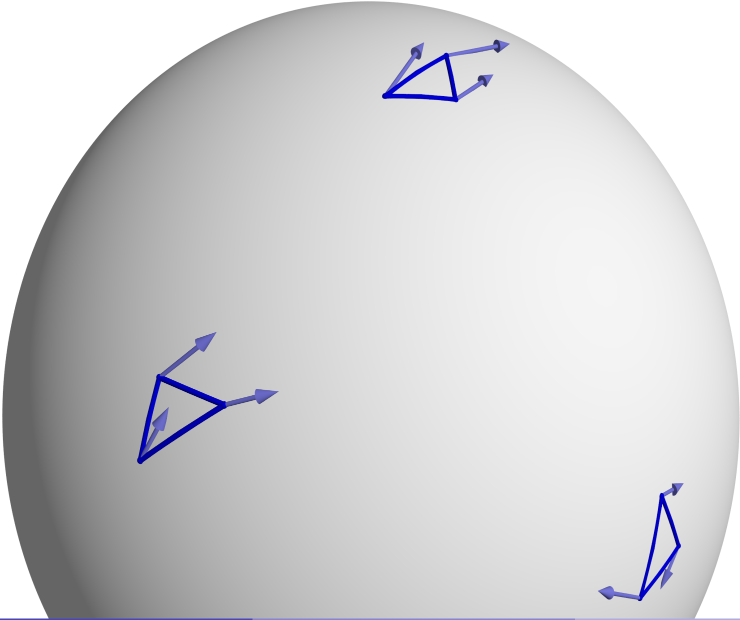
Hermite subdivision in $\mathrm{SO}(3)$



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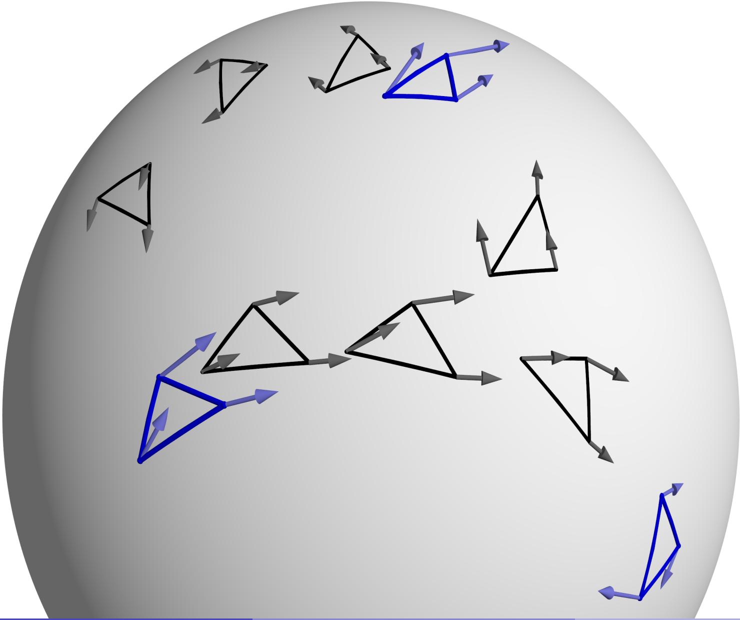

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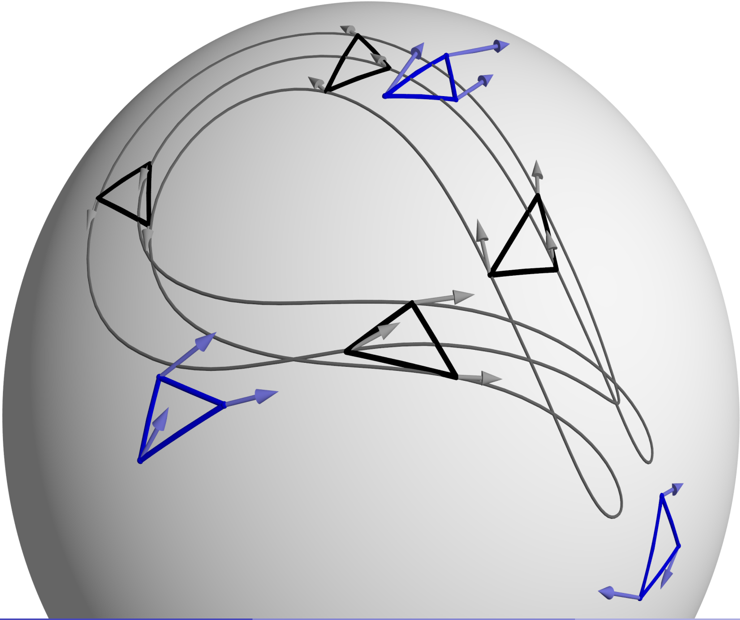

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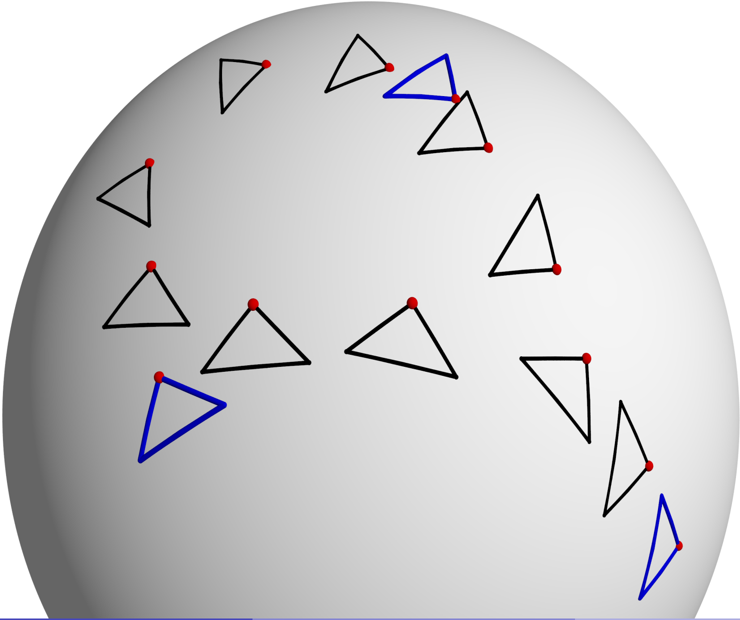

C^1 convergence analysis

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Thank you!