Linear multi-scale transforms based on reverse subdivision schemes

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Outline of the Talk

1. Multi-scale transforms based on interpolatory subdivision schemes

2. Reverse subdivision schemes

3. Multi-scale transforms based on reverse subdivision schemes

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Multi-scale transforms based on interpolatory subdivision schemes

Few names: Donoho + Yu, Oswald, Amat in $\mathbb{R}$

Donoho et.el., Grohs + Wallner for Manifold-valued data

Given data at refinemernt level J

$$\mathbf{f}^J = \{f_i^J \mid i \in \mathbb{Z}\}$$

with f_i^J related to the point $2^{-J}i$, and an interpolatory subdivision scheme S,

$$(S\mathbf{f})_{2i} = f_i, \quad (S\mathbf{f})_{2i+1} = F(f_{i-n}, \cdots, f_{i+n})$$

The multi-scale transform (MST) consists of a decomposition stage and a reconstruction stage.

Decomposition: from level j to level $j-1$ for $j = J, J-1, \cdots, 1$

$$f_i^{j-1} = f_{2i}^j, \quad i \in \mathbb{Z}, \text{ or shortly } \ \mathbf{f}^{j-1} = (\mathbf{f}^j \downarrow 2)$$

$$d_i^j = f_i^j - (S\mathbf{f}^{j-1})_i, \quad i \in \mathbb{Z}$$

By definition $d_{2i}^j = 0$, $i \in \mathbb{Z}$ or shortly $(\mathbf{d}^j \downarrow 2) = \mathbf{0}$

This process can be written shortly

$$\mathbf{f}^{j-1} = (\mathbf{f}^j \downarrow 2), \ \mathbf{d}^j = \mathbf{f}^j - S\mathbf{f}^{j-1}, \ j = J, \cdots, 1$$

Reconstruction: from level j to level $j+1$ for $j = 0, 1, \cdots, J-1$,

$$\mathbf{f}^{j+1} = S\mathbf{f}^j + \mathbf{d}^{j+1}.$$

Important to note:

- Knowledge of $\mathbf{f}^j$, and $\mathbf{d}^{j+1}$ is needed to compute $\mathbf{f}^{j+1}$.

- The amount of elements (number of elements related to a unit interval) in $\mathbf{f}^j$, together with that of the non-zero elements in $\mathbf{d}^{j+1}$ is equal to the amount of elements in $\mathbf{f}^{j+1}$.

- The sequences $\mathbf{f}^0, \mathbf{d}^j,\ j = 1, 2, \cdots, J$ are termed the pyramid generated by the MST.

- The elements of the pyramid determine $\mathbf{f}^J$

- The amount of elements in the pyramid equals that in $\mathbf{f}^J$.

- The MST has perfect reconstuction

Important properties of the Multi-scale transform when S is linear

• Decay with j of details when $\mathbf{f}^J$ is sampled from a smooth function f, $\mathbf{f}^J = f|_{2^{-J}\mathbb{Z}}$

$$\|\mathbf{d}^j\|_\infty \leq C2^{-j}$$

• Stability of Decomposition

There exists a constant D depending on the decomposition, such that for any $\mathbf{f}^J$ and $\tilde{\mathbf{f}}^J$, the two pyramids obtained by the decomposion

$$\mathbf{f}^0, \mathbf{d}^j,\ j = 1, \cdots, J, \quad \text{and} \quad \tilde{\mathbf{f}}^0, \tilde{\mathbf{d}}^j,\ j = 1, \cdots, J$$

satisfy

$$\max\{\|\mathbf{f}^0 - \tilde{\mathbf{f}}^0\|_\infty,\ |\mathbf{d}^j - \tilde{\mathbf{d}}^j\|_\infty,\ j = 1, \cdots, J\} \leq D\|\mathbf{f}^J - \tilde{\mathbf{f}}^J\|_\infty$$

- Stability of Reconstruction

There exists a constant R depending on the reconstruction, such that for any two pyramids

$$\mathbf{f}^0,\ \mathbf{d}^j,\ j = 1, \cdots, J \quad \text{and} \quad \tilde{\mathbf{f}}^0,\ \tilde{\mathbf{d}}^j,\ j = 1, \cdots, J,$$

the two sequences $\mathbf{f}^J$ and $\tilde{\mathbf{f}}^J$, obtained from these pyramids by the reconstruction, satisfy

$$\|\mathbf{f}^J - \tilde{\mathbf{f}}^J\|_\infty \leq R \max\{\|\mathbf{f}^0 - \tilde{\mathbf{f}}^0\|_\infty,\ \|\mathbf{d}^j - \tilde{\mathbf{d}}^j\|_\infty,\ j = 1, \cdots, J\}$$

How to construct a multi-scale transform with perfect reconstruction and small sparse details based on any linear subdivision scheme?

The difficulty is in the decomposition. How to get $\mathbf{f}^{j-1}$ from $\mathbf{f}^{j}$ such that $\mathbf{f}^{j} - S\mathbf{f}^{j-1}$ is small?

This is termed in the literature "reverse subdivision"

The methods known in the literature for "reverse subdivision" for MST, are based on least-squares fit, namely on constructing an operator A such that for a subdivision operator S and a sequence $\mathbf{f}$

$$A\mathbf{f} = \mathbf{g} \quad \text{if } \|\mathbf{f} - S\mathbf{g}\|_{\ell_2} \to \min$$

Main authors: Samavati, Bartels, Sadeghi

There are many problems with this approach. One clear disadvantage is that the sequence of details at any refinement level is not sparse

Our approach is to mimic the structure of the MST based on an interpolatory subdivision

(i) To reverse the subdivision **exactly** for the even elements of a sequence

(ii) To have the even elements of the details $\mathbf{d}^j = \mathbf{f}^j - S\mathbf{f}^{j-1}$ vanish

Goal (ii) is a direct consequence of goal (i)

The way to achieve goal (i) is to find an operator A for the first part of the decomposition

$$\mathbf{f}^{j-1} = A\mathbf{f}^j$$

so that

$$(SA\mathbf{f}^j \downarrow 2) = (\mathbf{f}^j \downarrow 2)$$

In what follows we assume that the subdivision operator S has a mask of finite support

$$(S\mathbf{f})_i = \sum_{\ell\in\mathbb{Z}} a_{i-2\ell} f_\ell, \quad \text{with } a_i = 0 \text{ for } i \notin [-n, n]$$

To "reverse subdevide" the sequence of the even elements in a given sequence, we define the *even symbol* of the subdivision S

$$a_e(z) = \sum_{\ell\in\mathbb{Z}} a_{2\ell} z^\ell$$

It follows from the theory of Wiener Algebra that if $t(z) = (a_e(z))^{-1}$ exists, then the Toeplitz matrix T with symbol $t(z)$ satisfies

$$(ST\mathbf{f} \downarrow 2) = (\mathbf{f} \downarrow 2)$$

A sufficient condition for the existence of the inverse symbol $t(z)$ is that the symbol $a_e(z)$ does not vanish for z on the unit circle.

We term subdivision schemes with even symbols satisfying the above sufficient condition *even-reversible*

Trivially all linear interpolatory subdivision schemes are even-reversible, since their even symbols equal 1.

We proved that

- All spline schemes are even-reversible.

- All pseudo-spline schemes, primal and dual, are even-reversible

Spline schemes play a central role in curve design. A multi-scale transform based on a spline scheme might have advantage in processing curves.

Properties of MST based on even-reversible subdivision schemes

- Decay rate of details:

For $\mathbf{f}^J = f|_{2^{-J}\mathbb{Z}}$, with $f \in C^1(\mathbb{R})$

$$\|\mathbf{d}^j\|_\infty \leq C\|T\|_\infty^{J-j} 2^{-j}, \quad j = 1, \cdots, J$$

In case S is interpolatory T is the identity and $\|T\| = 1$. Otherwise $\|T\|_\infty > 1$

- Stability of Reconstruction

- "Conditional" stability of Decomposistion

$$\|\mathbf{f}^0 - \tilde{\mathbf{f}}^0\|_\infty \leq \|T\|_\infty^J \|\mathbf{f}^J - \tilde{\mathbf{f}}^J\|_\infty$$

$$\|\mathbf{d}^j - \tilde{\mathbf{d}}^j\|_\infty \leq \ M\|T\|_\infty^{J-j}\|\mathbf{f}^J - \tilde{\mathbf{f}}^J\|_\infty, \quad j = 1, \cdots, J$$