Prony's method in several variables, oligonomials and Linear Algebra

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FORWISS

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The History

Introduction

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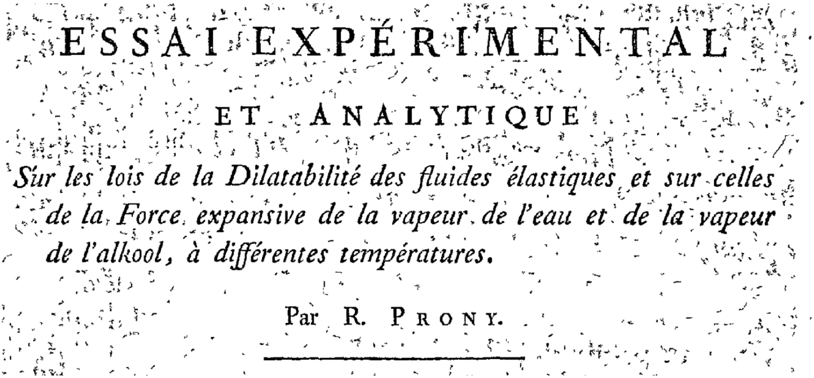
ESSAI EXPÉRIMENTAL

ET ANALYTIQUE

Sur les lois de la Dilatabilité des fluides élastiques et sur celles de la Force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures.

Par R. Prony.

The Problem

Prony's problem

For a *exponential polynomial*

$$f(x) = \sum_{\omega \in \Omega} f_\omega \, e^{\omega^T x}, \qquad \Omega \subset (\mathbb{R} + i\mathbb{R}/2\pi\mathbb{Z})^s, \quad 0 \neq f_\omega \in \mathbb{C},$$

determine Ω and f_ω from samples of f.

Oligonomials

For an *algebraic polynomial*

$$f(x) = \sum_{\alpha \in A} f_\alpha \, x^\alpha, \qquad A \subset \mathbb{N}_0^s, \, 0 \neq f_\alpha \in \mathbb{C}, \qquad \#A < \infty,$$

determine A and f_α from samples of f.

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For a sparse *exponential polynomial*

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Oligonomials (fewnomials)

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Remarks on Prony's Problem

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Assumptions

1. Problem is *sparse*: $\#\Omega / \#A$ small.
2. Frequencies $\omega \in \Omega$ and powers $\alpha \in A$ can vary.
3. No embedding in $\Pi_{\deg A}$.
4. A priory knowledge: $\#\Omega$ or $\#A$.

Problem structure (Prony)

1. Frequencies: *nonlinear* problem.
2. Coefficients: *linear* problem.
3. Evaluation points: Grids $\Gamma \subset \mathbb{Z}^s$.
4. Real only – complex straightforward.

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Polynomials

1. $\Pi = \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_s]$.
2. $\Pi_n = \left\{ p(x) = \sum_{|\alpha| \le n} p_\alpha x^\alpha : p_\alpha \in \mathbb{R} \right\}$ of degree n.
3. *Degree*

$$\deg p = \max \{ |\alpha| : f_\alpha \neq 0 \},$$

Coefficient vectors

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Prony's Trick

A Hankel matrix

$$F_n := \left[f(\alpha+\beta) : \begin{array}{l} |\alpha| \le n \\ |\beta| \le n \end{array} \right] \in \mathbb{R}^{d_n \times d_n}, \qquad d_n = \binom{n+s}{s}.$$

A computation ...

For $p \in \Pi_n$:

Consequence

$p(X_\Omega) = 0$ implies $F_n p = 0$.

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For $p \in \Pi_n$ and $|\alpha| \le n$:

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The ideal

1. $I_\Omega = I(X_\Omega) = \{p \in \Pi : p(X_\Omega) = 0\}$.
2. Zero dimensional ideal.
3. ...

Prony again

1. $p \in I_\Omega \cap \Pi_n \Rightarrow p \in \ker F_n$.
2. Converse?
3. $\Omega = \{\omega, \omega'\}, f_\omega = -f_{\omega'}$

Theorem

For n large enough we have $I_\Omega \cap \Pi_n \simeq \ker F_n$.

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Minimal Degree Interpolation

Goal

Answer the question *What is "large enough"?*

Interpolation space

$\mathscr{P} \subset \Pi$ *degree reducing interpolation space* with respect to a *finite* $X \subset \mathbb{R}^s$: for $g \in \Pi$ there exists a *unique* $p \in \mathscr{P}$ such that

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Ideals and Bases

H–bases

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2. Alternative: *homogeneous ideals*. (Macaulay 1916)

Definition

(Homogeneous) leading form

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(Homogeneous) leading form and *ideal generated by G*

$$\Lambda(p) = p^0_{\deg p} = \sum_{|\alpha| = \deg p} p_\alpha \, (\cdot)^\alpha, \qquad \langle G \rangle = \left\{ \sum_{g \in G} q_g \, g : q_g \in \Pi \right\}.$$

Reduction

Further ingredient

1. Inner product $(\cdot,\cdot) : \Pi \times \Pi \to \mathbb{R}$.
2. For example $(p,q) = \boldsymbol{p}^T \boldsymbol{q} = \sum_\alpha p_\alpha q_\alpha$.

Division with remainder for p, G

1. While $p \neq 0$
   1. Decompose $\Lambda(p) = \sum_{g \in G} p_g \Lambda(g) + r$
   2. Replace $p \leftarrow p - \sum_{g \in G} p_g g - r$.
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Reduction II

Theorem

If H is an *H–basis* and $p = \sum_{g \in G} p_h\, h + r$ computed by reduction, then

1. $r = 0$ iff $p \in \langle H \rangle$.
2. r depends only on $\langle H \rangle$

Definition

1. $\nu_I(p) := r = p - \sum_{g \in G} p_h\, h$ *normal form* of p modulo $\langle H \rangle =: I$.
2. $N_I := \nu_I(\Pi) \simeq \Pi / I$ *inverse system* for I. (Macaulay '16, Gröbner '37)

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Theorem (reduction is interpolation)

$N_{I(X)}$ is a degree reducing interpolation space for X.

The Normal Form Space

Normal forms for points

1. Given $X \subset \mathbb{R}^s$, $\#X < \infty$.
2. Basis for normal form space
3. Satisfies $\ell_x(x') = \delta_{xx'}$, $x, x' \in X$.

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$$\ell_x := \nu_{I(X)} \left(\prod_{x' \in X \setminus \{x\}} \frac{(x - x')(\cdot - x')}{\|x - x'\|^2} \right), \qquad x \in X.$$

3. Satisfies $\ell_x(x') = \delta_{xx'}$, $x, x' \in X$.

Theorem

For $p \in \Pi$ are equivalent:

1. p is reduced, i.e. $p = \nu_{I(X)}(p)$.
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Normal forms for points

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Goal

1. Compute H–basis and normal form space N_Ω for $I_\Omega = I(e^\Omega)$.
2. Compute frequencies Ω (standard).
3. Compute coefficients f_ω (linear system).

Slightly different Hankel matrix

$$F_{n,k} = \left[f(\alpha+\beta) : \begin{array}{c} |\alpha| \le n \\ |\beta| \le k \end{array} \right], \qquad k \le n.$$

The "magic" number

1. Fundamental guess: $n > d(X_\Omega) = \deg N_\Omega$, e.g., $n \ge \#\Omega$.
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Linear Algebra I

Kernel vs. rank

With $F_{n,k} = U_k \Sigma V_k^T$:

1. $\ker F_{n,k}$: V_* columns of V for $\sigma_{\ell,\ell} = 0$.
2. $\operatorname{rank} F_{n,k}$: V_+ columns wrto nonzero singular values.

The H–basis update

1. With
$$H_{\le k-1} = [H_0 \dots H_{k-1}], \qquad H_j = [h_{j1} \dots h_{jv_j}],$$
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$$V_*^T H_{\le k-1} = QR = [Q_1\, Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}.$$
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Linear Algebra II

The normal forms

1. General idea: $N_k = \Pi_k^0 \ominus \Lambda(H_k)$.
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$$\Lambda(H_k) = Q_k \begin{bmatrix} R_k \\ 0 \end{bmatrix}, \qquad Q_k = [Q_{k,1} \; Q_{k,2}].$$
3. $N_j = Q_{j,2}$.

Reduction

1. With $k := \deg p$ solve $R_k c = Q_{k,1}^T \Lambda(p)$.
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The normal forms

1. General idea: $N_k = \Pi_k^0 \ominus \Lambda(H_k)$.
2. Again
$$\Lambda(H_k) = Q_k \begin{bmatrix} R_k \\ 0 \end{bmatrix}, \qquad Q_k = [Q_{k,1} \; Q_{k,2}].$$
3. $N_j = Q_{j,2}$.

Reduction

1. With $k := \deg p$ solve $R_k c = Q_{k,1}^T \Lambda(p)$.
2. Set $r_k^0 = \Lambda(p) - \Lambda(H_k)\, c$.
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Ideals and Normal Forms

Theorem

If $k > \deg N_\Omega$

1. The columns of $H_{\leq k}$ are coefficient vectors of an H–basis of I_Ω.
2. The matrix $N_{\leq k}$ contains a graded homogeneous basis for N_Ω.

Remarks

1. All by means of standard Linear Algebra (`Matlab`).
2. H–basis is *very* redundant.
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Multiplication tables

1. Multiplication: $N_\Omega \ni p$
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Observation (reduction is interpolation)

$$\nu(p) = \sum_{\omega \in \Omega} p(x_\omega)\, \ell_\omega$$

Theorem (Stetter, Sticklberger, …)

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3. Eigenvalue computation and matching …

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$$L_{k,j} N_k, \qquad L_{k,j} = \sum_{|\alpha|=k} e_{\alpha+\epsilon_j} e_\alpha^T.$$

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$$\left(I - Q_{k,1} Q_{k,1}^T \right) L_{k,j} N_k, \qquad L_{k,j} = \sum_{|\alpha|=k} e_{\alpha+\epsilon_j} e_\alpha^T.$$

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The Linear System

Solve . . .

$$\left[e^{\omega^T \alpha} : \begin{array}{c} |\alpha| \le k \\ \omega \in \Omega \end{array} \right] [f_\omega : \omega \in \Omega] = [f(\alpha) : |\alpha| \le k] .$$

Properties

1. Transpose of *Vandermonde matrix* for X_Ω.
2. Full rank.
3. Overdetermined system, stabilizes.
4. Right hand side: part of $F_{n,0}$.

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"Complete" Algorithm

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Procedure

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3. Compute multiplication tables.
4. Determine and match eigenvalues.
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1. Surprisingly easy to implement in `Matlab/Octave`.
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Some Results

Random frequencies & coefficients, real, 100 tests

parameters			average error		max error	
s	# freq.	n	coeff	freq	coeff	freq
2	5	3	1.3688e-11	1.8332e-09	3.5131e-09	2.4165e-07
2	10	5	4.9366e-08	2.6388e-06	7.3010e-05	5.3330e-04
2	15	8	7.0614e-07	2.9725e-04	1.4659e-04	4.4493e-02
2	20	9	Inf	Inf	NaN	NaN
3	20	6	1.5874e-08	1.4165e-06	4.7337e-05	8.9382e-04
4	20	5	8.4712e-12	4.6565e-11	9.0309e-09	3.7456e-09
5	20	5	1.6879e-12	5.9416e-11	1.9510e-09	1.3243e-08
5	50	5	1.1079e-10	6.6070e-10	3.1709e-07	6.6913e-08
5	100	6	2.9307e-09	1.9431e-08	1.0034e-05	1.3912e-06
5	150	8	1.3142e-08	8.4199e-08	5.7281e-06	4.3975e-06

Some Results

Random frequencies, purely imaginary, 100 tests

parameters			average error		max error	
s	# freq.	n	coeff	freq	coeff	freq
2	10	5	1.3476e-14	3.4744e-13	6.0290e-12	1.3724e-10
2	20	7	2.5148e-14	1.2420e-12	3.2103e-11	7.8847e-10
2	50	11	5.9357e-14	3.9721e-12	1.1845e-10	5.5214e-09
2	100	15	9.0480e-13	5.7684e-11	8.8308e-09	2.0468e-07
5	100	6	2.3796e-15	4.3794e-15	3.1431e-11	3.2918e-14
5	150	8	2.3954e-15	4.7773e-15	1.1702e-11	6.9726e-14

Some Results

Line through origin, top real, bottom imaginary, 100 tests

parameters			average error		max error	
s	# freq.	fail	coeff	freq	coeff	freq
2	3	0	5.1668e-06	1.6241e-04	0.0023195	0.0243550
2	4	0	2.8912e-06	2.9318e-03	9.9505e-04	5.8547e-01
2	5	5	3.1901e-05	2.1641e-02	0.0058405	1.8753920
2	10	100	∅	∅	∅	∅
3	5	5	1.4484e-05	2.8744e-02	0.0016677	1.5547492
3	4	14	2.1197e-05	1.2439e-01	4.2678e-03	1.8590e+01
3	5	24	3.8699e-04	3.9617e-02	0.057250	1.326782
2	5	0	2.1330e-12	3.6481e-11	3.9225e-10	4.5345e-09
2	10	0	3.1867e-06	5.9222e-03	0.0018326	1.7269961
2	20	11	8.0145e-06	4.2270e-03	0.0071972	1.0316399
3	10	1	0.0025437	0.0206981	1.0404	4.3874

Oligonomials

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Recall

Sparse polynomials

$$f(x) = \sum_{\alpha \in A} f_\alpha \, x^\alpha.$$

$O\lambda\iota\gamma o-$ $\simeq$ "few"

Simple transformation

1. Pick any invertible $\Xi \in \mathbb{C}^{s \times s}$.
2. Consider

$$F_{n,k}(\Xi) := \left[e^{\Xi(\beta+\gamma)} : \begin{array}{l} |\beta| \le n \\ |\gamma| \le k \end{array} \right].$$

3. Prony situation with $\Omega = \Xi^T A = \{\Xi^T \alpha : \alpha \in A\}$;
4. Round to next integer: $A = [\Xi^{-T} \Omega]$.

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