Approximation order of non-stationary subdivision schemes

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Outline

The stationary 1D case: review of known results

Definition (Approximation order)

Let $\gamma \in \mathbb{N}$, $f \in C^\gamma(\mathbb{R})$ with $\|f^{(\gamma)}\|_\infty < \infty$. A convergent, stationary subdivision scheme $\{S_\mathbf{a}\}$ is said to have **approximation order** γ if the limit function $g_{\mathbf{f}^{[0]}} := S_\mathbf{a}^\infty \mathbf{f}^{[0]}$ obtained from $\mathbf{f}^{[0]} = \{f(ih),\ i \in \mathbb{Z}\}$, $h \in \mathbb{R}_+$ is such that

$$\|g_{\mathbf{f}^{[0]}} - f\| \leq C_f\, h^\gamma$$

with C_f a positive constant depending only on f.

Theorem [de Boor (1990)]

$\{S_\mathbf{a}\}$ has approximation order γ if it reproduces the space $\Pi_{\gamma-1}$ of polynomials of degree $d \leq \gamma - 1$, i.e. if it satisfies $g_{\mathbf{f}^{[0]}} = f$ for all initial sequences $\mathbf{f}^{[0]} = \{f(i+p),\ i \in \mathbb{Z}\}$ where $f \in \Pi_{\gamma-1}$ and $p \in \mathbb{R}$.

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☞ Proof based on the Taylor expansion of f and the fact that the basic limit function $\phi := S_{\mathbf{a}}^{\infty}\,\delta$ is finitely supported.

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The non-stationary 1D case: preliminary definitions

Definition (Basic limit function)

Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a convergent, non-stationary subdivision scheme. For $\boldsymbol{\delta} := \{\delta_{i,0},\ i \in \mathbb{Z}\}$, we call

$$\phi_m := \lim_{\ell \to \infty} S_{\mathbf{a}^{[m+\ell]}} \cdots S_{\mathbf{a}^{[m]}} \boldsymbol{\delta}, \qquad m \geq 0$$

the family of **basic limit functions** of $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$.

For $\mathbf{f}^{[m]} = S_{\mathbf{a}^{[m-1]}} \cdots S_{\mathbf{a}^{[1]}} S_{\mathbf{a}^{[0]}} \mathbf{f}^{[0]}$, we can write the **limit** of the subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ applied to the data $\mathbf{f}^{[m]} = \{\mathrm{f}_i^{[m]}, i \in \mathbb{Z}\}$, $m \geq 0$, as

$$g_{\mathbf{f}^{[m]}} = \lim_{\ell \to \infty} S_{\mathbf{a}^{[m+\ell]}} \cdots S_{\mathbf{a}^{[m]}} \mathbf{f}^{[m]} = \sum_{i \in \mathbb{Z}} \mathrm{f}_i^{[m]} \phi_m(2^m \cdot - i)$$

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Let Ω be a compact set in $\mathbb{R}$.

Definition (Sobolev space)

$W_\infty^\rho(\Omega) := \{f \in L_\infty(\Omega) \ : \ f^{(\ell)} \in L_\infty(\Omega) \text{ for all } 0 \le \ell \le \rho\}, \quad \rho \in \mathbb{N}$

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Let $\{S_{\mathbf{a}^{[k]}},\ k \ge 0\}$ be a convergent, non-stationary subdivision scheme and denote by $g_{\mathbf{f}^{[0]}}$ the limit function obtained from the initial data $\mathbf{f}^{[0]}$.
We say that $\{S_{\mathbf{a}^{[k]}},\ k \ge 0\}$ has **approximation order** γ if

$$\|g_{\mathbf{f}^{[0]}} - f\|_{L_\infty(\Omega)} \le C_f\, h^\gamma,$$

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Reproduction of exponential polynomials

Let $\eta \in \mathbb{N}$. Assume $\lambda_n \in \mathbb{C}$, $\mu_n \in \mathbb{N}$ for all $n = 1, \cdots, \eta$ and define:

- $N := \sum_{n=1}^{\eta} \mu_n$
- $\Phi_N := \mathrm{span}\{x^\beta e^{\lambda_n x}, \beta = 0, \cdots, \mu_n - 1, n = 1, \cdots, \eta\}$, $x \in \mathbb{R}$

Definition (Φ_N-generation / Φ_N-reproduction)

Let $f \in \Phi_N$ and let $t_i^{[0]}$, $i \in \mathbb{Z}$ be ordered equidistant values on the real axis. A convergent, non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ is called

i) **Φ_N-generating** :
if for all initial sequences $\mathbf{f}^{[0]} = \{f(t_i^{[0]}),\ i \in \mathbb{Z}\}$ it provides $g_{\mathbf{f}^{[0]}} \in \Phi_N$;

ii) **Φ_N-reproducing** :
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How to check reproduction of Φ_N?

Proposition [Conti and R. (2011)]

Let $a^{[k]}(z) = \sum_{i\in\mathbb{Z}} \mathrm{a}_i^{[k]} z^i$, $z \in \mathbb{C}\setminus\{0\}$ be the kth level symbol of a convergent and non-singular subdivision scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$. Then $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ reproduces Φ_N w.r.t. the parametrization $T^{[k]} = \{t_i^{[k]} = \frac{i+p}{2^k},\ i \in \mathbb{Z}\}$ (with shift parameter $p \in \mathbb{R}$) if and only if

(a)
$$a^{[k]}\left(-e^{-\frac{\lambda_n}{2^{k+1}}}\right) = 0, \qquad n = 1, \dots, \eta$$
$$\frac{d^\beta}{dz^\beta} a^{[k]}\left(-e^{-\frac{\lambda_n}{2^{k+1}}}\right) = 0,\ \beta = 1, \dots, \mu_n - 1$$

(b)
$$a^{[k]}\left(e^{-\frac{\lambda_n}{2^{k+1}}}\right) = 2\left(e^{-\frac{\lambda_n}{2^{k+1}}}\right)^p, \qquad n = 1, \dots, \eta$$
$$\frac{d^\beta}{dz^\beta} a^{[k]}\left(e^{-\frac{\lambda_n}{2^{k+1}}}\right) = 2\left(e^{-\frac{\lambda_n}{2^{k+1}}}\right)^{p-\beta} \prod_{j=0}^{\beta-1}(p-j),\ \beta = 1, \dots, \mu_n - 1$$

Asymptotical similarity versus asymptotical equivalence

Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a non-stationary subdivision scheme with subdivision masks $\{\mathbf{a}^{[k]},\ k \geq 0\}$ and let $\{S_{\mathbf{a}}\}$ be a stationary subdivision scheme with subdivision mask $\{\mathbf{a}\}$.

☞ Hereinafter we always assume $\mathbf{a}^{[k]}$, $k \geq 0$ and $\mathbf{a}$ finitely supported.

Definition (Asymptotical equivalence - Dyn and Levin (1995))

$\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ and $\{S_{\mathbf{a}}\}$ are termed **asymptotically equivalent** if

$$\mathrm{supp}(\mathbf{a}^{[k]}) = \mathrm{supp}(\mathbf{a}) \text{ for all } k \geq 0 \text{ and } \sum_{k=0}^{\infty} \|\mathbf{a}^{[k]} - \mathbf{a}\|_\infty < \infty.$$

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Properties of basic limit functions I

Proposition A [Conti, R. and Yoon (2016)]

Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a Φ_1-reproducing non-stationary subdivision scheme which is *asymptotically similar* to a convergent, stationary subdivision scheme $\{S_{\mathbf{a}}\}$ with stable basic limit function of Hölder continuity $\alpha \in (0,1)$.
Then the associated basic limit functions $\{\phi_m,\ m \geq 0\}$ and ϕ satisfy

$$\lim_{m\to\infty} \|\phi_m - \phi\|_\infty = 0.$$

In view of the fact that ϕ is bounded we obtain

Corollary

$\{\phi_m,\ m \geq 0\}$ is uniformly bounded independently of m, i.e.

$$\|\phi_m\|_\infty \leq M \quad \forall m \geq 0$$

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Approximation order result I

Theorem I [Conti, R. and Yoon (2016)]

Let $\Phi_\gamma := \langle \varphi_0, ..., \varphi_{\gamma-1} \rangle$, $\gamma \in \mathbb{N}$. Assume that the non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ is Φ_γ-reproducing and *asymptotically similar* to a convergent, stationary subdivision scheme $\{S_{\mathbf{a}}\}$ with stable basic limit function of Hölder continuity $\alpha \in (0,1)$. Assume further that the initial data are of the form $\mathbf{f}^{[m]} := \{\mathrm{f}_i^{[m]} = f(2^{-m}i),\ i \in \mathbb{Z}\}$ for some fixed $m \geq 0$ and for some function $f \in W_\infty^\gamma(\Omega)$. If the Wronskian matrix $\mathcal{W}_{\Phi_\gamma}(0) := \left(\frac{\varphi_s^{(r)}(0)}{r!},\ r, s = 0, ..., \gamma - 1 \right)$ is invertible, then

$$\|g_{\mathbf{f}^{[m]}} - f\|_{L_\infty(\Omega)} \leq C_f\, 2^{-\gamma m}, \quad m \geq 0$$

with a constant $C_f > 0$ depending only on f.

Properties of basic limit functions II

Proposition B

Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a non-stationary subdivision scheme with subdivision masks $\{\mathbf{a}^{[k]},\ k \geq 0\}$ and let $\{S_{\mathbf{a}}\}$ be a convergent, stationary subdivision scheme with subdivision mask $\{\mathbf{a}\}$. If $\|\mathbf{a}^{[k]} - \mathbf{a}\|_\infty \leq C\,2^{-\nu k}$ with $\nu \in \mathbb{N}$, then the associated basic limit functions $\{\phi_m,\ m \geq 0\}$ and ϕ satisfy

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Approximation order result II

Theorem II

Let $\{S_{\mathbf{a}}\}$ be a convergent, stationary subdivision scheme reproducing Π_{N-1}. Let $\Phi_\gamma := \langle \varphi_0, \dots, \varphi_{\gamma-1} \rangle$ with $\gamma \in \mathbb{N}$, $\gamma \le N$ and assume that the non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}},\ k \ge 0\}$ is Φ_γ-reproducing. Assume further that the corresponding subdivision masks $\{\mathbf{a}^{[k]},\ k \ge 0\}$ and $\{\mathbf{a}\}$ satisfy $\|\mathbf{a}^{[k]} - \mathbf{a}\|_\infty \le C\, 2^{-\nu k}$ with some $\nu \in \mathbb{N}$. If the Wronskian matrix $\mathcal{W}_{\Phi_\gamma}(0) := \left(\frac{\varphi_s^{(r)}(0)}{r!},\ r, s = 0, \dots, \gamma - 1 \right)$ is invertible and the initial data are of the form $\mathbf{f}^{[m]} := \{\mathrm{f}_i^{[m]} = f(2^{-m} i),\ i \in \mathbb{Z}\}$ for some fixed $m \ge 0$ and for some function $f \in W_\infty^N(\Omega)$, then

$$\|g_{\mathbf{f}^{[m]}} - f\|_{L_\infty(\Omega)} \le C_f\, 2^{-\sigma m}, \quad m \ge 0$$

with $\boxed{\sigma = \min(\gamma + \nu, N)}$ and C_f a positive constant depending only on f.

Sketch of the proofs of Theorems I and II

Common steps:

- Let $x \in \Omega$ and let $\mathbf{f} = (f^{(r)}(x),\ r = 0, ..., \gamma - 1)^T$. Denote by $\mathbf{d} = (d_n,\ n = 0, ..., \gamma - 1)^T$ the unique solution of $\mathcal{W}_{\Phi_\gamma}(0)\,\mathbf{d} = \mathbf{f}$
- Define $\psi := \psi_x := \sum_{n=0}^{\gamma-1} d_n \, \varphi_n(\cdot - x)$

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- $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ is Φ_γ-reproducing $\Rightarrow$ $\psi = \sum_{i \in \mathbb{Z}} \psi(2^{-m} i)\ \phi_m(2^m \cdot - i)$

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- $f(x) = \psi(x) \Rightarrow$

$$f(x) - g_{\mathbf{f}^{[m]}}(x) = \sum_{i\in\mathbb{Z}} \left(\psi(2^{-m}i) - f(2^{-m}i)\right) \phi_m(2^m x - i)$$

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Proof of Theorem I

- For $f \in W_\infty^\gamma(\Omega)$, we write the degree-$(\gamma-1)$ Taylor expansion (T.E.) of $\psi - f$ around x as

$$\psi(2^{-m}i)-f(2^{-m}i)=\sum_{r=0}^{\gamma-1}\frac{(2^{-m}i-x)^r}{r!}(\psi-f)^{(r)}(x)+\frac{(2^{-m}i-x)^\gamma}{\gamma!}(\psi-f)^{(\gamma)}(\xi_i)$$

for some ξ_i between x and $2^{-m}i$

- $(\psi-f)^{(r)}(x)=0,\ r=0,\dots,\gamma-1 \ \Rightarrow$

$$f(x)-g_{\mathbf{f}^{[m]}}(x)=2^{-\gamma m}\sum_{i\in\mathbb{Z}}\phi_m(2^m x-i)\,(i-2^m x)^\gamma\,\frac{(\psi-f)^{(\gamma)}(\xi_i)}{\gamma!}$$

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for some ξ_i between x and $2^{-m}i$

- $(\psi - f)^{(r)}(x) = 0,\ r = 0, ..., \gamma - 1 \ \Rightarrow$

$$f(x) - g_{\mathbf{f}[m]}(x) = 2^{-\gamma m} \sum_{i\in\mathbb{Z}} \phi_m(2^m x - i)\,(i - 2^m x)^\gamma\, \frac{(\psi - f)^{(\gamma)}(\xi_i)}{\gamma!}$$

- $|\psi^{(\gamma)}(\xi_i)| \leq C\|f\|_{W_\infty^\gamma(\Omega)}$, ϕ_m compactly supported and uniformly bounded independently of m (Prop.A) $\Rightarrow$ $|f(x) - g_{\mathbf{f}[m]}(x)| \leq C_f\, 2^{-\gamma m}$ □

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$$f(x) - g_{\mathbf{f}^{[m]}}(x) = 2^{-\gamma m} \sum_{i\in\mathbb{Z}} \phi_m(2^m x - i)\,(i - 2^m x)^\gamma \,\frac{(\psi - f)^{(\gamma)}(\xi_i)}{\gamma!}$$

- $|\psi^{(\gamma)}(\xi_i)| \leq C\|f\|_{W_\infty^\gamma(\Omega)}$, ϕ_m compactly supported and uniformly bounded independently of m (Prop.A) $\Rightarrow \ |f(x) - g_{\mathbf{f}^{[m]}}(x)| \leq C_f\, 2^{-\gamma m}$ □

Proof of Theorem II

- For $f \in W_\infty^N(\Omega)$, we write the degree-$(N-1)$ T.E. of $\psi - f$ around x:

$$\psi(2^{-m}i) - f(2^{-m}i) = \sum_{r=0}^{N-1} \frac{(2^{-m}i - x)^r}{r!} (\psi - f)^{(r)}(x) + \frac{(2^{-m}i - x)^N}{N!} (\psi - f)^{(N)}(\xi_i)$$

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$$\Big|\sum_{i\in\mathbb{Z}}\phi_m(2^m x - i)\,(2^{-m}i - x)^r\Big| \le 2^{-mr}\sum_{i\in\mathbb{Z}}\Big|\phi_m(2^m x - i) - \phi(2^m x - i)\Big|\,|i - 2^m x|^r$$

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- Conditions for checking Φ_γ-reproduction [Charina, Conti and R. (2014)]
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Explanation:

such d provides the highest possible degree of the polynomial space Π_d^2 s.t.

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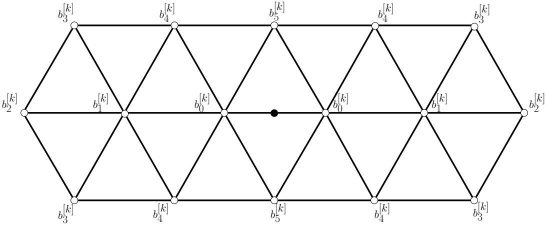
A bivariate example [Novara, R. and Yoon (2016)]

We consider the interpolatory scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ with edge point stencil:



a) $\lambda \in [0, \pi) \cup \mathrm{i}\mathbb{R}^+$
$v^{[k]} = \cos\left(\frac{\lambda}{2^{k+1}}\right), \; \forall k \geq 0$

b) $w^{[k]} \to w$ with the rate of $O(2^{-2k})$ as $k \to \infty$

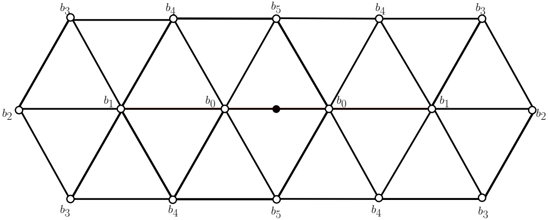
$$
\begin{aligned}
b_0^{[k]} &= 2(2(v^{[k]})^2 - 1)w^{[k]} + \frac{(2v^{[k]}+1)^2}{8v^{[k]}(v^{[k]}+1)} \\
b_1^{[k]} &= -(4(v^{[k]})^2 - 1)w^{[k]} - \frac{1}{8v^{[k]}(v^{[k]}+1)} \\
b_2^{[k]} &= w^{[k]}
\end{aligned}
$$

$$
\begin{aligned}
b_3^{[k]} &= -(2(v^{[k]})^2 - 1)w^{[k]} + \frac{2v^{[k]}+1}{64(v^{[k]})^2(2v^{[k]}-1)(v^{[k]}+1)^2} \\
b_4^{[k]} &= 4(v^{[k]})^2(2(v^{[k]})^2 - 1)w^{(k)} - \frac{2v^{[k]}+1}{16(2v^{[k]}-1)(v^{[k]}+1)^2} \\
b_5^{[k]} &= -2(4(v^{[k]})^2 - 1)(2(v^{[k]})^2 - 1)w^{[k]} + \frac{(2v^{[k]}+1)^2}{32(v^{[k]})^2(v^{[k]}+1)^2}
\end{aligned}
$$

☞ For all choices of $\{w^{[k]}, \; k \geq 0\}$ in b), $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is Φ_γ-reproducing with

$$\Phi_\gamma = \{1, x, y, e^{\pm\lambda x}, e^{\pm\lambda y}, e^{\pm\lambda(x+y)}, e^{\pm\lambda(x-y)}\}, \quad \gamma = 11 > \sharp(\Pi_3^2)$$

$\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is *asymptotically equivalent* to the interpolatory stationary scheme $\{S_{\mathbf{a}}\}$ having edge point stencil



$$b_0 = 2w + \frac{9}{16} \qquad b_1 = -3w - \frac{1}{16} \qquad b_2 = w$$
$$b_3 = \frac{3}{256} - w \qquad b_4 = 4w - \frac{3}{64} \qquad b_5 = \frac{9}{128} - 6w$$

☞ Indeed, the associated subdivision masks $\{\mathbf{a}^{[k]}, k \geq 0\}$ and $\{\mathbf{a}\}$ satisfy

$$\|\mathbf{a}^{[k]} - \mathbf{a}\|_\infty \leq C\, 2^{-2k}$$

☞ The stationary scheme $\{S_{\mathbf{a}}\}$ reproduces $\boxed{\Pi_5^2}$ for all $w \in \mathbb{R}$

☞ Since $N = 6$, $\nu = 2$, $d = 3$, in view of Theorem II(2D), for $f \in W_\infty^6(\Omega)$ the scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ has **approximation order** 6.

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Thank you for your attention!

Approximation order of non-stationary subdivision schemes

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