Fractional and Complex Pseudo-Splines

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Outline

- Pseudo-Splines
- Fractional and Complex Pseudo-Splines
- Lowpass Properties of Complex Pseudo-Splines
- Relation to Known Approaches and Outlook

Pseudo-Splines of Integer Order (m, ℓ)

Introduced by Daubechies, Han, Ron and Shen.

Motivation: Construct families of refinable functions which interpolate between the classical B-splines ($\ell = 0$) and the interpolatory refinable functions of Dubuc's ($\ell = m - 1$).

Focus: Construction of framelets for $L^2(\mathbb{R})$ with required approximation order.

Filters (Type II):

$$H_0^{(m,\ell)}(\gamma) := (\cos^2 \pi\gamma)^m \sum_{k=0}^{\ell} \binom{m+\ell}{k} (\sin^2 \pi\gamma)^k (\cos^2 \pi\gamma)^{\ell-k}, \gamma \in \mathbb{R}.$$

Why Fractional and Complex Order?

1. Increased flexibility with regard to smoothness: A *discrete* family of functions from C^{m-1}, $m \in \mathbb{N}$, is replaced by a *continuous* family of functions from Hölder spaces $C^{\alpha-1}$.

2. The presence of the imaginary part of z allows for direct utilization in complex transform techniques for signal and image analyses.

3. Unlike the classical Schoenberg polynomial splines, which allow the construction of Parseval wavelet frames for $L^2(\mathbb{R})$ via the unitary or oblique extension principle, the fractional and complex B-splines cannot in general be used for this type of construction.

Parseval Wavelet Frames

D: unitary dilation operator $(Df)(x) := \sqrt{2}f(2x)$.

T_k: translation operator $(T_k f)(x) := f(x-k)$, $k \in \mathbb{Z}$.

Parseval wavelet frames of the form

$\{D^j T_k \psi_0\}_{j,k\in\mathbb{Z}} \cup \cdots \cup \{D^j T_k \psi_n\}_{j,k\in\mathbb{Z}}$ and $\{\psi_1, \ldots, \psi_n\} \subset L^2(\mathbb{R})$

Construct functions ψ_l such that

$$\sum_{l=1}^{n} \sum_{j,k\in\mathbb{Z}} |\langle f, D^j T_k \psi_l \rangle|^2 = \|f\|^2, \quad \forall f \in L^2(\mathbb{R}),$$

or, equivalently,

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, D^j T_k \psi_l \rangle D^j T_k \psi_l, \quad \forall f \in L^2(\mathbb{R}).$$

Polynomial B-Splines of even order m generate wavelet frames.

Fractional or complex B-Splines do ***not***.

"Reason:" For $\gamma \in \mathbb{T} := (-\frac{1}{2}, \frac{1}{2})$

$$1 = (\cos^2 \pi\gamma + \sin^2 \pi\gamma)^m \quad \Longrightarrow \quad 1 = (\cos^2 \pi\gamma + \sin^2 \pi\gamma)^z$$

For $z \in \mathbb{N}$, the r.h.s. splits for $z \in \mathbb{C} \setminus \mathbb{N}$, $\operatorname{Re} z > 1$, into two parts:

(1) $\sum_{k=0}^{\infty} \binom{z}{k} (\sin^2 \pi\gamma)^k \, (\cos^2 \pi\gamma)^{z-k}$ converges for $\gamma \in (-\frac{1}{4}, \frac{1}{4})$.

(2) $\sum_{k=0}^{\infty} \binom{z}{k} (\cos^2 \pi\gamma)^k \, (\sin^2 \pi\gamma)^{z-k}$ converges for $\gamma \in \mathbb{T} \setminus (-\frac{1}{4}, \frac{1}{4})$.

Pseudo-Splines of Fractional and Complex Order

For $\gamma \in \mathbb{R}$ consider the filters:

$$H_0(\gamma) := H_0^{(z,\ell)}(\gamma) := (\cos^2 \pi\gamma)^z \sum_{k=0}^{\ell} \binom{z+\ell}{k} (\sin^2 \pi\gamma)^k (\cos^2 \pi\gamma)^{\ell-k},$$

where $z \in \mathbb{C}$ with $\alpha := \operatorname{Re} z \geq 1$ and $0 \leq \ell \leq \lfloor \alpha \rfloor - 1$.

Theorem (Christensen, Forster, M. 2016)

Let $z \in \mathbb{C}_{\geq 1}$, and let $\ell = 0, 1, \ldots, \lfloor \alpha \rfloor - 1$. Then

$$0 < \vartheta \leq |H_0(\gamma)|^2 + \left|H_0\left(\gamma + \tfrac{1}{2}\right)\right|^2 \leq 1, \quad \forall \gamma \in \mathbb{T},$$

and some positive constant $\vartheta = \vartheta(z, \ell)$.

The Cascade Algorithm

We are seeking a refinable function $\varphi \in L^2(\mathbb{R})$ associated with the filter $H_0 \in L^2(\mathbb{T})$, i.e., a function such that

$$\widehat{\varphi}(\gamma) = H_0\left(\tfrac{\gamma}{2}\right)\widehat{\varphi}\left(\tfrac{\gamma}{2}\right), \qquad \gamma \in \mathbb{R}.$$

As in classical wavelet analysis we identify an appropriate function φ via the cascade algorithm:

$$\widehat{\varphi}(\gamma) = \prod_{m=1}^{\infty} H_0(2^{-m}\gamma)\widehat{\varphi}(0), \quad \gamma \in \mathbb{R}.$$

Thus, we define the functions φ_m, $m \in \mathbb{N}_0$, via

$$\widehat{\varphi}_0(\gamma) := \chi_{[-\frac{1}{2},\frac{1}{2}]}(\gamma),$$

$$\widehat{\varphi}_m(\gamma) := \chi_{[-2^{m-1},2^{m-1}]}(\gamma)\prod_{j=1}^{m} H_0(2^{-j}\gamma).$$

Theorem (Christensen, Forster, M. 2016)

Let $H_0(\gamma)$ be a 1-periodic real or complex function satisfying the following conditions:

(i) $H_0(0) = 1$.

(ii) *There exist a constant $C > 0$ and an exponent $\varepsilon > 0$ with*

$$|H_0(\gamma) - 1| \le C \cdot |\gamma|^{\varepsilon}, \quad \forall\, \gamma \in \mathbb{R}.$$

(iii) *There exists a positive constant ϑ such that*

$$0 < \vartheta \le |H_0(\gamma)|^2 + |H_0(\gamma + \tfrac{1}{2})|^2 \le 1, \quad \forall\, \gamma \in \mathbb{R}.$$

Then $\{\widehat{\varphi}_m\}$ converges pointwise and uniformly on compact subsets. The pointwise limit $\widehat{\varphi} \in L^2(\mathbb{R})$ and $\varphi_m \to \varphi$ in $L^2(\mathbb{R})$. Furthermore, φ satisfies the above refinement equation.

Refinable Functions Associated With H_0

Theorem (Christensen, Forster, M. 2016)

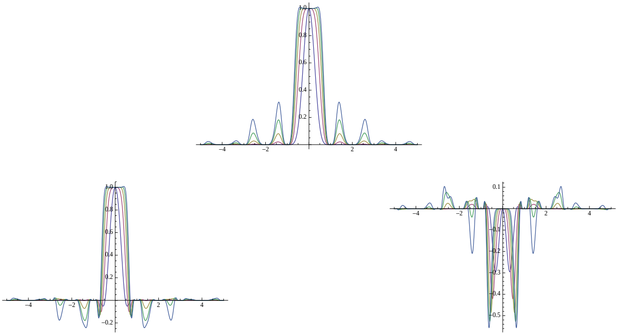
The filter

$$H_0(\gamma) = H_0^{(z,\ell)}(\gamma) = (\cos^2 \pi\gamma)^z \sum_{k=0}^{\ell} \binom{z+\ell}{k} (\sin^2 \pi\gamma)^k (\cos^2 \pi\gamma)^{\ell-k},$$

where $z \in \mathbb{C}$ *with* $\alpha := Re\, z \geq 1$ *and* $0 \leq \ell \leq \lfloor \alpha \rfloor - 1$, *generates a refinable function* φ *via the cascade algorithm.*

We call the function φ a *pseudo-spline of complex order* (z, ℓ) or for short a *complex pseudo-spline*.

Pseudo-Splines in the Fourier Domain



Pseudosplines $\widehat{\varphi}^{(\sqrt{5}+2i,\ell)}$ for $\ell = 0, 1, \ldots, 4$:
$|\widehat{\varphi}^{(\sqrt{5}+2i,\ell)}|$ (above), $\operatorname{Re}\widehat{\varphi}^{(\sqrt{5}+2i,\ell)}$ (left), $\operatorname{Im}\widehat{\varphi}^{(\sqrt{5}+2i,\ell)}$ (right).

The pseudospline parameter ℓ allows the tuning of the width of the lowpass property of the refinable function φ.

Lowpass Properties of Complex Pseudo-Splines

- $0 < \vartheta \leq |H_0(\gamma)|^2 \leq 1, \, \forall \, \gamma \in \mathbb{R}$.
- Consequentially, $|\widehat{\varphi}(\gamma)| \leq 1, \, \forall \, \gamma \in \mathbb{R}$.
- As expected for a refinable function, the pseudo-splines act as lowpass filters. In fact, there exists a neighborhood of the origin, where $\widehat{\varphi}$ does not vanish.

Set $H_0^{(z,\ell)}(\gamma) := H_0^{(z,0)}(\gamma) P^{(z,\ell)}(\gamma)$, where

$$P^{(z,\ell)}(\gamma) := \sum_{k=0}^{\ell} \binom{z+\ell}{k} (\sin^2 \pi\gamma)^k \, (\cos^2 \pi\gamma)^{\ell-k}.$$

Theorem (Christensen, Forster, M. 2016)

Suppose $z := x + iy$, $x \geq 1$, *is such that*

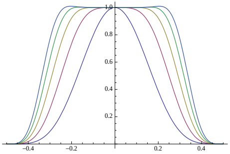
$$\sum_{j=0}^{\ell} \tan^{-1} \frac{y}{x+j} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad (*)$$

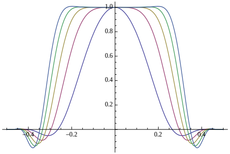
holds. Then there exists a positive constant $c > 0$ *that bounds* $\widehat{\varphi}$ *from below in a neighborhood of the origin, i.e.,*

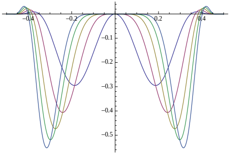
$$0 < c \leq |\widehat{\varphi}^{(z,0)}(\gamma)| \leq |\widehat{\varphi}^{(z,\ell)}(\gamma)|.$$

For $\ell = 0$, condition $(*)$ is satisfied, as expected, since $H^{(z,0)}$ is the filter of the classical fractional or complex B-spline.

Pseudo-Spline Filters







Filters $H_0^{(\sqrt{5}+2i,\ell)}$ for $\ell = 0, 1, \ldots, 4$.

$|H_0^{(\sqrt{5}+2i,\ell)}|$ (above), $\mathrm{Re}\, H_0^{(\sqrt{5}+2i,\ell)}$ (left), $\mathrm{Im}\, H^{(\sqrt{5}+2i,\ell)}$ (right).

The pseudo-spline parameter ℓ allows the width of H_0 to be tuned. The imaginary part acts as an added bandpass filter.

Relation to Known Approaches and Outlook

Pseudo-splines with parameter $\ell = 0$ are the symmetric fractional B-splines $\beta_*^{2\alpha-1}$.

For complex z with $\operatorname{Re} z \geq 1$ and $\ell = 0$ they correspond to the complex B-splines β_y^{2z-1} with shift $y = 0$.

As a variant, one could consider complex pseudo-splines with shift, namely

$$H_0^{(z,\ell,y)}(\gamma) := \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\frac{z+1}{2}-y} \left(\frac{1-e^{i\omega}}{-i\omega}\right)^{\frac{z+1}{2}+y} \times \sum_{k=0}^{\ell} \binom{z+\ell}{k} (\sin^2 \pi\gamma)^k (\cos^2 \pi\gamma)^{\ell-k},$$

for $y, z \in \mathbb{C}$, $\operatorname{Re} z \geq 1$ and $0 \leq \ell \leq \lfloor \operatorname{Re} z \rfloor - 1$. The shift y may allow for better adaption to the signal or image, as was shown in the case of symmetric complex B-splines.

Pseudo-splines $H_0^{(m,m-1)}$, $m \in \mathbb{N}$, are related to Dubuc's interpolatory refinable functions.

- Do we get with our method, a new variant of fractional and complex interpolating fractional splines?
- Are there fractional or complex subdivision schemes?
- For the fractional case, we have symmetric filters. Do the fractional pseudo-splines interpolate the subdivision schemes of Dubuc's which are also symmetric?

THANK YOU!