Almansi Formula on the Sphere and New Cubature Formulas with Error Bounds

Polyharmonic Paradigm

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Signals, Images, and Approximation, Bernried, March 2016

Alexander von Humboldt
Stiftung / Foundation

The speaker‘s attendance at this conference was sponsored by the Alexander von Humboldt Foundation.

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- **G. G. Hardy, A Mathematician's Apology**

Polyharmonic Paradigm - what is it ?

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$$\Delta^N u(x) = 0 \qquad \text{in domain } D$$

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- Generalize the $1D$ **odd-degree polynomials** $P_{2N-1}(t)$ by Hermite interpolation: they solve the **Boundary Value problem**
$$\begin{aligned} \frac{d^{2N}}{dt^{2N}} P_{2N-1}(t) &= 0 \\ \frac{d^j}{dt^j} P_{2N-1}(0) &= c_j \qquad \text{for } j = 0, 1, ..., N-1 \\ \frac{d^j}{dt^j} P_{2N-1}(1) &= d_j \qquad \text{for } j = 0, 1, ..., N-1 \end{aligned}$$

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- In multivariate case - solution of Boundary Value problem:

$$\Delta^N u(x) = 0 \qquad \text{in domain } D$$

$$\Delta^j u(x) = g_j(x) \qquad \text{for } x \in \partial D$$

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- Multivariate Moment problem, Operator theory (as Tensor representations), and multivariate Quadrature (= **Cubature**) - the present talk

One-dimensional reminder on quadrature formulas

- The $N-$point Quadrature formula of Gauss:

$$\int_{-1}^{1} f(t)\,dt \approx \sum_{j=1}^{N} \lambda_j f(t_j) = G_N[f]$$

$$-1 < t_j < 1,\ \lambda_j > 0,$$

exact for polynomials f with $\deg f \leq 2N-1$;

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- If $P_N(t)$ is the orthogonal polynomial of degree N i.e.

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- For the polynomials $P_N(t)$ – $3-$term recurrence relations which reduces the computation of the knots t_j to a simple and fast Linear Algebra.
- **IMPORTANT**: There are **Error bounds** for the Gauss-Jacobi formula

Jacobi's point of view

- **F**or a weight function $w(t) > 0$ (originally, $w(t) = (1+t)^{\alpha}(1-t)^{\beta}$) let
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where $P_N(t_j) = 0$ and $\lambda_j > 0$, but
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- **Example**: Compute
$$\int_0^1 g(t)\,\frac{1}{\sqrt{t}}dt$$
for a polynomial $g(t)$ in two ways: using **Gauss** G_N, or using **Gauss-Jacobi** GJ_N for $w(t) = \frac{1}{\sqrt{t}}$.

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- There is **no Jacobi's** point of view!
- Cubature formulas –Orthogonal polynomials of several variables by Hermite 1889, Appelle, Radon, Sobolev, etc.
- Solve equations for finding λ_j and x_j – problems with **error bounds**.

Cubature formulas

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- Following Jacobi's **point of view**, assume that $f(x)$ has representation

$$f(x) = P(x)\, w(x)$$

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- How to proceed?
- We need a new point of view on the multivariate polynomials.

Our approach in the 2D case – in the disc B

- We consider the Fourier expansions for general functions P and w, where $z = x + iy$:

$$P(z) = \sum_{k=-\infty}^{\infty} p_k(r) e^{ik\varphi} \qquad z = re^{i\varphi}, \ r = |z|$$

$$w(z) = \sum_{k=-\infty}^{\infty} w_k(r) e^{ik\varphi}$$

and

$$p_k(r) := \frac{1}{2\pi} \int_0^{2\pi} P\left(re^{i\varphi}\right) e^{-ik\varphi} d\varphi; \quad w_k(r) := \frac{1}{2\pi} \int_0^{2\pi} w\left(re^{i\varphi}\right) e^{-ik\varphi}$$

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- Hence, $\int_B f(z)\, dz$ is reduced to

$$I := \int_B P(z) w(z)\, dz = 2\pi \sum_{k=-\infty}^{\infty} \int_0^1 p_k(r) w_{-k}(r)\, rdr$$

A remarkable representation of multivariate polynomials, 2D case

- Let $P(x,y)$ be a polynomial in $\mathbb{R}^2$ satisfying $\Delta^N P(x,y) = 0$. Then the following **remarkable Almansi** representation holds

$$P(x,y) = \sum_{k=-\infty}^{\infty} \widetilde{p}_k\left(r^2\right) r^k e^{ik\varphi} \qquad z = r \times e^{i\varphi},\ r = |z|$$

where $\widetilde{p}_k$ is a $1D$ polynomial of degree $\leq N-1$.

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- Hence, the **polyharmonic degree** N (in Δ^N) is a generalization for the one-dimensional degree N of the polynomials.
- This is a fundamental point of the so-called **Polyharmonic Paradigm.**

The integral as infinite sum of 1-dim integrals

Hence, for polynomials $P(x)$ we obtain for $\rho = r^2$

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- Here the new weigth $\widetilde{w}_{k,\ell}(\rho)$ is defined by

$$\widetilde{w}_k(\rho)\, d\rho := r^k w_k(r)\, r dr = \frac{1}{2} \rho^{\frac{k}{2}} w_k\left(\sqrt{\rho}\right) d\rho$$

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- For every $k \in \mathbb{Z}$ and $N \geq 1$, we apply $N-$point Gauss-Jacobi quadrature:

$$\int_0^1 p_k(\rho)\, \widetilde{w}_k(\rho)\, d\rho \approx \sum_{j=1}^{N} p_k\left(t_{j;k}\right) \lambda_{j;k}$$

which is exact for polynomials p_k satisfying $\deg p_k \leq 2N-1$.

The cubature formula defined:

Now, let $g(x)$ be a continuous function with Fourier expansion

$$g(x) = \sum_k g_k(r)\, e^{ik\varphi}$$

The integral becomes

$$\begin{aligned} \int_B g(z)\, w(z)\, dz &= \sum_k \int_0^1 g_k(r)\, w_k(r)\, rdr \\ &= \frac{1}{2} \sum_k \int_0^1 g_k(\sqrt{\rho})\, \rho^{-\frac{k}{2}} \rho^{\frac{k}{2}} w_k(\sqrt{\rho})\, d\rho \\ &\approx \frac{1}{2} \sum_k \sum_{j=1}^N g_k\left(\sqrt{t_{j;k}}\right) t_{j;k}^{-\frac{k}{2}} \times \lambda_{j;k} \\ &=: C(g) \end{aligned}$$

The miracle - Chebyshev inequality applied

- Important to see convergence of $C(g)$, i.e.:

$$2C(g) = \sum_{k} \sum_{j=1}^{N} g_k \left(\sqrt{t_{j;k}} \right) \cdot t_{j;k}^{-\frac{k}{2}} \cdot \lambda_{j;k} < \infty.$$

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$$\left| \sum_{j=1}^{N} g_k\left(t_{j;k}\right) \cdot t_{j;k}^{-\frac{k}{2}} \cdot \lambda_{j;k} \right| \leq C \left\| g \right\|_{\sup} \int w_k\left(\sqrt{\rho}\right) d\rho$$

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$$\left| \sum_{j=1}^{N} g_k\left(t_{j;k}\right) \cdot t_{j;k}^{-\frac{k}{2}} \cdot \lambda_{j;k} \right| \leq C \|g\|_{\sup} \int w_k\left(\sqrt{\rho}\right) d\rho$$

- Further, we impose the condition

$$\|w\| := \sum_{k,\ell} \int w_k\left(\sqrt{\rho}\right) d\rho < \infty$$

Final approximation of the Fourier coefficients

To finish the Cubature formula, approximate the coefficients $g_k(r)$.
In $\mathbb{R}^2$ we have have

$$g_k(r) = \frac{1}{2\pi}\int_0^{2\pi} g\left(re^{i\varphi}\right)e^{ik\varphi}d\varphi$$

Hence, for integers $M \geq 1$, the approximation is just the **trapezoidal rule**:

$$f_k^{(M)}(r) := \frac{2\pi}{M}\sum_{s=1}^{M} f\left(re^{i\frac{2\pi s}{M}}\right)e^{i\frac{2\pi s}{M}}$$

For real-valued functions g, the **final Cubature formula** is:

$$\int_B g(z)\,w(z)\,dz \approx \frac{\pi}{M}\sum_{k=0}^{K}\sum_{j=1}^{N}\sum_{s=1}^{M}\lambda_{j,k}\cdot t_{j,k}^{-\frac{k}{2}}\cdot e^{i\frac{2\pi s}{M}}\cdot g\left(\sqrt{t_{j,k}}e^{i\frac{2\pi s}{M}}\right)$$

Exactness space

- The knots are

$$\sqrt{t_{j,k}}e^{i\frac{2\pi s}{M}} \qquad 0 \leq s \leq M-1,\ |k| \leq K,\ j = 1, ..., N$$

and the weights are

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- The formula is **exact** for the polynomials

$$P(x,y) = r^{2s} r^k e^{ik\varphi} = |z|^{2s} z^k$$

  for $0 \le s \le 2N-1$; $0 \le k \le M-1-K$

Nice properties of the Cubature formula – stability estimate

The coefficients satisfy the stability estimate

$$\left| \frac{\pi}{M} \sum_{k=0}^{K} \sum_{j=1}^{N} \sum_{s=1}^{M} \lambda_{j,k} \cdot t_{j,k}^{-\frac{k}{2}} \cdot e^{i\frac{2\pi s}{M}} \right| \leq C_1 \|w\| .$$

By a theorem of Polya and others, we have a stable Cubature formula.

Error estimates

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- Details are available in **arxiv**: **http://arxiv.org/abs/1509.00283**

Almansi formula for the spherical harmonics on the sphere

- On the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ we consider the integral

$$I_w(f) = \int_{\mathbb{S}^2} f(\Theta)\, w(\Theta)\, d\sigma_\Theta \qquad \text{with } d\sigma_\Theta = \sin\vartheta\, d\varphi\, d\vartheta$$

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- For $\Theta \in \mathbb{S}^2$ we have the representation, with $0 \leq \varphi < 2\pi$ and $0 \leq \vartheta < \pi$,

$$\Theta_1 = \sin\vartheta\cos\varphi, \ \Theta_2 = \sin\vartheta\sin\varphi, \ \Theta_3 = \cos\vartheta$$

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- We put $x = \cos\vartheta,\ \vartheta = \arccos x$. Then, for $k = 0, 1, ...; \quad |\lambda| \leq k$, the spherical harmonics $\left\{Y_k^\lambda(\Theta)\right\}$ on $\mathbb{S}^2$ are normalized as:

$$Y_k^\lambda(\Theta) = N_{k,\lambda} \times e^{i\lambda\varphi} P_k^\lambda(\cos\vartheta) = N_{k,\lambda} \times e^{i\lambda\varphi}\left(1 - x^2\right)^{\frac{|\lambda|}{2}} P_k^{(|\lambda|)}(x)$$

where P_k^λ are the **associated Legendre polynomials, and** P_k are the usual. Recall that

$$\deg P_k^\lambda = k - |\lambda|.$$

Almansi type formula

- Since every harmonic polynomial $P(x)$ on $\mathbb{S}^2$ (or even restriction of an arbitrary polynomial to $\mathbb{S}^2$) is representable by means of spherical harmonics; see **Stein-Weiss** book.

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- For $\Theta \in \mathbb{S}^2$ we obtain

$$\begin{aligned} P(\Theta) &= \sum_{k=0}^{K} \sum_{\lambda=-k}^{k} \alpha_{k,\lambda} Y_k^{\lambda}(\Theta) \\ &= \sum_{k=0}^{K} \sum_{\lambda=-k}^{k} \alpha_{k,\lambda} \left(N_{k,\lambda} e^{i\lambda\varphi} \left(1-x^2\right)^{\frac{|\lambda|}{2}} P_k^{(|\lambda|)}(x) \right) \\ &= \sum_{\lambda=-\infty}^{\infty} e^{i\lambda\varphi} \left(1-x^2\right)^{\frac{|\lambda|}{2}} p_{\lambda}(x), \end{aligned}$$

where $p_{\lambda}(x)$ are polynomials.

Reduced integral

- On the other hand, we have the infinite sum of $1D$ integrals:

$$I_w(f) = \int_{\mathbb{S}^2} f(\Theta)\, w(\Theta)\, d\sigma_\Theta = \sum_{k=-\infty}^{\infty} \int_0^{\pi} f_k(\cos\vartheta)\, w_{-k}(\cos\vartheta) \sin\vartheta d\vartheta$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \int_{-1}^{1} f_k(x)\, w_{-k}(x)\, dx;$$

here f_k and w_k are the Fourier coefficients, e.g.

$$f_k(x) := \frac{1}{2\pi} \int_0^{2\pi} f(\Theta)\, e^{-ik\varphi} d\varphi$$

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- Hence, if $f = P$ is a polynomial, we obtain

$$\int_{-1}^{1} f_\lambda(x)\, w_{-\lambda}(x)\, dx = \int_{-1}^{1} \left(1 - x^2\right)^{\frac{|\lambda|}{2}} p_\lambda(x)\, w_{-\lambda}(x)\, dx$$

$$= \int_{-1}^{1} p_\lambda(x)\, \widetilde{w}_{-\lambda}(x)\, dx, \quad \text{where } \widetilde{w}_\lambda(x) = \left(1 - x^2\right)^{\frac{|\lambda|}{2}} w_\lambda(x)$$

Details

Will appear in a book:

- O. Kounchev, H. Render, The Multidimensional Moment problem, Hardy spaces, and Cubature formulas, in preparation for Springer

References

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- O. Kounchev, H. Render (2005), Reconsideration of the multivariate moment problem and a new method for approximating multivariate integrals; http://arxiv.org/pdf/math/0509380v1.pdf

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- O. Kounchev, H. Render (2010), The moment problem for pseudo-positive definite functionals. Arkiv for Matematik, vol. 48 : 97-120.

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- O. Kounchev, H. Render, 2015, A new cubature formula with weight functions on the disc, with error estimates; http://arxiv.org/abs/1509.00283

Some perspectives

- In subdivision on homogeneous spaces – use the theory of spherical harmonics of Harish-Chandra

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Some perspectives

- In subdivision on homogeneous spaces – use the theory of spherical harmonics of Harish-Chandra
- Wavelets on such as well.
- Moment and Cubature theories on such; e.g. on the Lorenz group the hypergeometric series is the analog to the spherical harmonics.