Continuous wavelet analysis in higher dimensions
An overview

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Wavelet analysis: Signal smoothness vs. Coefficient decay

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- Local version: fast coefficient decay away from the singularities.
- Global version: weighted summability of the coefficients characterizes function spaces of sparse signals (e.g., Besov spaces)

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The dual action is the key feature of the group.

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Unifying concept

The dual action is the key feature of the group.
Of particular importance: The blind spot of the wavelet transform.

Outline

1. Higher dimensional continuous wavelet transform

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2. Coorbit space theory

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2. Coorbit space theory
3. Constructing compactly supported atoms

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4. Decomposition space description of coorbit spaces

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General Setup: d-dimensional CWT

- Dilation group $H < \mathrm{GL}(d, \mathbb{R})$, a closed matrix group
- $G = \mathbb{R}^d \rtimes H$, the affine group generated by H and translations. As a set, $G = \mathbb{R}^d \times H$, with group law

$$(x, h)(y, g) = (x + hy, hg) \ .$$

- $\mathrm{L}^p(G)$ denotes L^p-space w.r.t. left Haar measure $d(x, h) = dx \frac{dh}{|\det(h)|}$.
- Quasi-regular representation of G acts on $\mathrm{L}^2(\mathbb{R}^d)$ via

$$(\pi(x, h)f)(y) = |\det(h)|^{-1/2} f(h^{-1}(y - x)) \ .$$

- Continuous wavelet transform: Given suitable $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ and $f \in \mathrm{L}^2(\mathbb{R}^d)$, let

$$\mathcal{W}_\psi f : G \to \mathbb{C} \ , \ \mathcal{W}_\psi f(x, h) = \langle f, \pi(x, h)\psi \rangle$$

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Fourier transform of the dilated wavelet $\pi(0,h)\psi$ is given by

$$\left(\pi(0,h)\psi\right)^\wedge(\xi) = |\mathrm{det}(h)|^{1/2}\psi(h^T\xi)\,.$$

This introduces the dual action

$$(h,\xi) \ni H \times \mathbb{R}^d \mapsto h^{-T}\xi \in \mathbb{R}^d\;.$$

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I.e. there exists $\xi_0 \in \mathbb{R}^d$ such that $\mathcal{O} = H^T\xi_0$ is open with complement of measure zero, and the stabilizer

$$H_{\xi_0} = \{h \in H : h^T\xi_0 = \xi_0\}$$

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is compact.
$\mathcal{O}^c := \mathbb{R}^d \setminus \mathcal{O}$ is called the blind spot of the wavelet transform.

Standing assumption and wavelet inversion

Theorem 1 (HF)

The standing assumption is equivalent to the property that π is a discrete series representation. In particular, there exist wavelets $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ such that

$$\mathcal{W}_\psi : \mathrm{L}^2(\mathbb{R}^n) \hookrightarrow \mathrm{L}^2(G)$$

is isometric. This equivalent to the (weak-sense) wavelet inversion formula

$$f = \int_G \mathcal{W}_\psi f(x,h) \; \pi(x,h)\psi \; d(x,h) \; .$$

Some examples in dimension two

1. Diagonal group:

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : ab \neq 0 \right\}$$

2. Similitude group:

$$H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 \neq 0 \right\}$$

3. Shearlet group(s):

$$H_c = \left\{ \pm \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} : a > 0 \right\} \quad (c \in \mathbb{R})$$

($c = 1/2$: Kutyniok/Labate/Dahlke/Steidl/Teschke ...)

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$$\|f\|_{CoY} = \|\mathcal{W}_\psi f\|_Y \quad .$$

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  - Discretization: When can the norm be expressed in terms of a discrete set of sampled wavelet coefficients?

Sample result from coorbit space theory

Let $1 \leq p < 2$. Assume that $\psi \in \mathcal{B}_{v_0}$, for the control weight

$$v_0(x,h) = w_0(h) = \max(1, \Delta_G(h)) .$$

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Then, for all suitably dense $\Gamma \subset \mathbb{R}^d$, $\Lambda \subset H$, the system

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Note: The coorbit space $Co(L^p)$ is independent of $\psi \in \mathcal{B}_{v_0}$.

Building blocks of coorbit theory: Frame atoms

Definition 2

Let $v_0 : G \to \mathbb{R}^+$ be continuous and submultiplicative. We call $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ v_0-atom if $\mathcal{W}_\psi \psi \in W(L^\infty, \mathrm{L}^1_{v_0})$, i.e., the function

$$G \ni (x,h) \mapsto \sup_{(y,g)\in U} |\mathcal{W}_\psi \psi\,((x,h)(y,g))| \in \mathbb{R}^+$$

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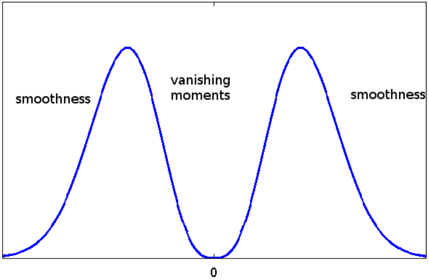
- Exhibit convenient subsets of $\mathcal{B}_{v_0}$.
- Are there compactly supported functions in $\mathcal{B}_{v_0}$? How do you construct them?

Outline

Reminder: Nice wavelets in dimension one

Desirable properties of wavelets

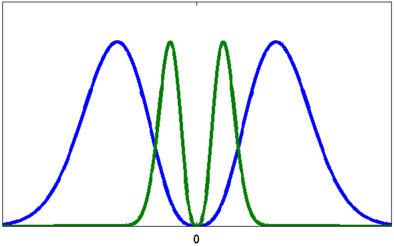
A nice wavelet $\psi \in \mathrm{L}^2(\mathbb{R})$ typically has three properties: Fast decay, smoothness, vanishing moments.
Concisely: Nice wavelets have good time-frequency localization.
(Note: Frequency-side localization is understood away from zero.)



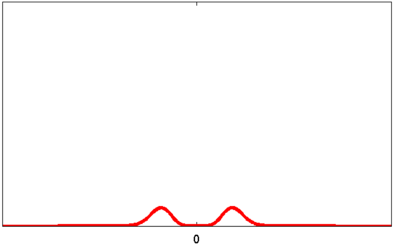
Vanishing moments and wavelet coefficient decay

Assumptions on nice wavelet ψ guarantee fast decay of $\mathcal{W}_\psi \psi$:

$$|\mathcal{W}_\psi \psi(x,s)| \le \left\| \partial^\ell \left(\widehat{\psi} \cdot \overline{\widehat{\psi}(s^{-1} \cdot)} \right) \right\|_1 |s|^{-1/2} (1+|x|)^{-\ell}$$



Plot of $\widehat{\psi}$ and $\widehat{\psi}(3\cdot)$



Overlap $\widehat{\psi} \cdot \widehat{\psi}(3\cdot)$

$\Rightarrow$ vanishing moments, smoothness govern decay of overlap, as $|s| \to 0, \infty$

Criteria for atoms

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Definition 3

Let $r \in \mathbb{N}$ be given. $f \in \mathrm{L}^1(\mathbb{R}^d)$ has vanishing moments in $\mathcal{O}^c$ of order r if all distributional derivatives $\partial^\alpha \widehat{f}$ with $|\alpha| < r$ are continuous functions, identically vanishing on $\mathcal{O}^c$.

Fourier envelope

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Definition 4

Let $\mathcal{O} \subset \mathbb{R}^d$ denote the dual orbit. Given $\xi \in \mathcal{O}$, let $\mathrm{dist}(\xi, \mathcal{O}^c)$ denote the euclidean distance of ξ to $\mathcal{O}^c$. Let

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Fourier envelope

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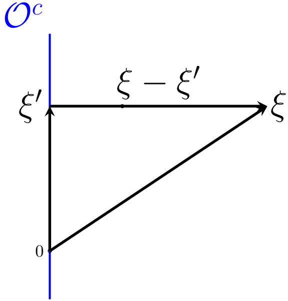

$$A(\xi) = \min \left(\frac{|\xi - \xi'|}{1 + |\xi'|}, \frac{1}{1+|\xi|} \right)$$

with ξ' = point in $\mathcal{O}^c$ closest to ξ

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Then any function ψ *with* $|\widehat{\psi}|_{r,r} < \infty$ *and vanishing moments in* $\mathcal{O}^c$ *of order* r *is in* $\mathcal{B}_{v_0}$.

Constructing compactly supported atoms

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- Clearly, picking $f \in C_c^k(\mathbb{R}^d)$, for k sufficiently large, is enough.

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- $\rightsquigarrow$ Theorem 5 is applicable, with

$$e_1 = d \ , \ e_2 = 1 \ , \ e_3 = d \ , \ e_4 = 0 \ .$$

- Resulting number of vanishing moments:

$$r = \left\lfloor \frac{d}{2} \right\rfloor + 6d + 3 \ .$$

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- Setup:

$$H = H_c = \left\{ \pm \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \; : \; a, b \in \mathbb{R}, a > 0 \right\} ,$$

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- In the classical shearlet case ($c = 1/2$), vanishing moments of order 19 suffice.

Verifying the conditions

Theorem 7 (HF/R. Raisi Tousi)

Assume that H fulfills the standing assumption, and belongs to one of the following classes:

- $H = \mathbb{R}^+ \cdot SO(d)$; *or*
- H *is abelian; or*
- H *is a generalized shearlet dilation group, i.e. there exists a closed abelian matrix group S consisting of unipotent matrices, the shearing subgroup), and a diagonal matrix Y generating the scaling subgroup such that*

$$H = \{\exp(rY)s : r \in \mathbb{R}, s \in S\} \ , \ or$$

- *any group constructed from the above using direct products and conjugation by arbitrary invertible matrices.*

Then H fulfills the conditions of Theorem 5, with explicitly computable exponents $e_1, \ldots, e_4$.

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An alternative description of wavelet coorbit spaces

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- Main idea: Uniformly cover the frequencies by a family of open sets. Decompose functions using a subordinate partition of unity. Introduce a norm by locally taking L^p-norms, and then globally combine using weighted ℓ^q.

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Theorem 8 (HF, F. Voigtlaender)

Every wavelet coorbit space $Co(L^{p,q}_v)$ is a decomposition space. The frequency covering underlying the latter is computed using the dual action.

Group H

Dual orbit $\mathcal{O}$

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$$\Big\downarrow \quad \begin{array}{l} p_{\xi_0} : H \to \mathcal{O}, h \mapsto h^T \xi_0 \\ \text{proper orbit map} \end{array}$$

Dual orbit $\mathcal{O}$

Group H $\begin{cases} (h_i)_{i\in I} \text{ well-spread in } H \\ \text{(continuous) weight } v : H \to (0,\infty) \end{cases}$

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Decomposition space norm

Fix a suitable partition of unity $(\varphi_i)_{i\in I}$ on $\mathcal{O}$ subordinate to $\mathcal{Q}$ and define

$$\|f\|_{\mathcal{D}\left(\mathcal{Q},L^p,\ell_u^q\right)} = \left\| \left(\left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_p \right)_{i\in I} \right\|_{\ell_u^q} = \left\| \left(u_i \cdot \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_p \right)_{i\in I} \right\|_{\ell^q}.$$

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  - $Co(L^{p,q}) \hookrightarrow W^{k,p}(\mathbb{R}^n)$.

Outline

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Definition 9 (Cones and truncated cones)

Given $\xi \in S^{d-1}$ and $\epsilon, R > 0$, we let

$$C(\xi,\epsilon) \quad = \quad \{r\xi' : \xi' \in S^{d-1} \cap B_\epsilon(\xi), r > 0\}$$

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or equivalently, for any $R > 0$:

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Definition 11

Let $\xi \in \mathcal{O} \cap S^{d-1}$, and $\epsilon, \delta, R > 0$ be such that $B_\delta(\xi_0) \subset \mathcal{O}$ and $C(\xi,\epsilon) \subset \mathcal{O}$. We define sets $K_i(\xi,\epsilon,\delta,R) \subset K_o(\xi,\epsilon,\delta,R) \subset H$ by

$$\begin{aligned} K_i(\xi,\epsilon,\delta,R) &= \{h \in H : h^{-T} B_\delta(\xi_0) \subset C(\xi,\epsilon,R)\} \\ K_o(\xi,\epsilon,\delta,R) &= \{h \in H : h^{-T} B_\delta(\xi_0) \cap C(\xi,\epsilon,R) \neq \emptyset\} \end{aligned}$$

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Meaning of the conditions

- Microlocal admissibility allows to use the matrix norm of h as scale parameter.
- The cone approximation property formalizes the ability of the wavelet system to distinguish more directions, as the scale goes to zero.

Generalized shearlet groups

Theorem 13 (HF/S. Dahlke/G. Alberti/F. DeMari/E. DeVito)

Let $H < \mathrm{GL}(d, \mathbb{R})$ be a generalized shearlet dilation group. Assume that the diagonal matrix Y generating the scaling subgroup of H has entries $(1, c_2, \ldots, c_d)$ with $0 < c_i < 1$, for $i = 2, \ldots, d$. Then the associated wavelet transform characterizes the wavefront set.

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- In higher dimensions, there are many fundamentally different shearlet dilation groups to which this theorem is applicable.

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