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Fakultät für Informatik und Mathematik

Rotational Anisotropic Wavelet Transform

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Motivation

- industry project with Micro-Epsilon GmbH & Co. KG
- medium-sized family-run company near Passau
- main focus on measurement technology

Motivation

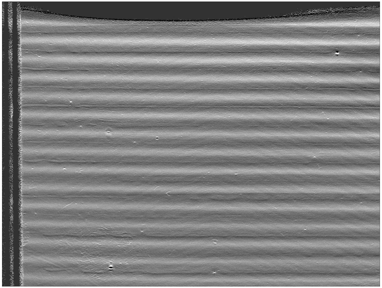


Figure: product surface with chatter marks

Problem

- in metal processing different cold rolls are used for producing metal bands with different thickness

Motivation

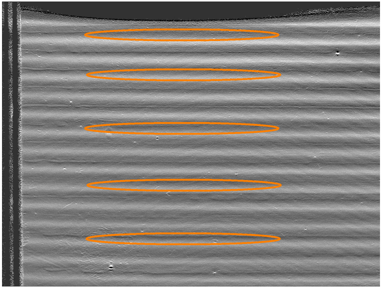


Figure: product surface with chatter marks

Problem

- in metal processing different cold rolls are used for producing metal bands with different thickness
- chatter marks occur when cold rolls are defect
- detect defect cold roll out of characteristics

Motivation

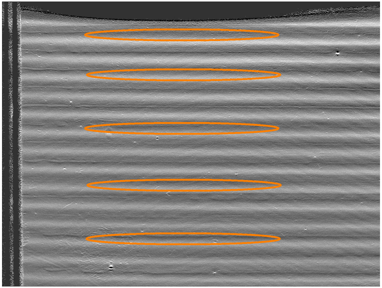
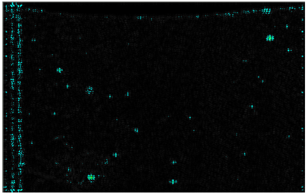


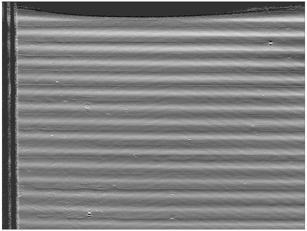
Figure: product surface with chatter marks

Aim

Detection of width and direction of chatter marks

First Approach





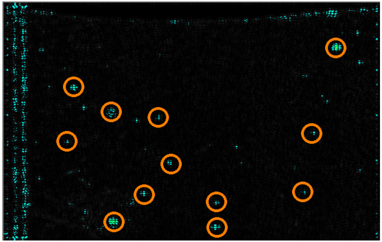
2D wavelet transform

- translation $b \in \mathbb{R}^2$
- dilation $a \in \mathbb{R} \setminus \{0\}$
- rotation $\theta \in [0, 2\pi)$

Pictures

- above: wavelet transform with $a = 3, \theta = 0$
- below: original image

First Approach



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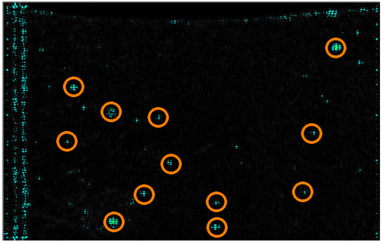
Picture

- wavelet transform with $a = 3, \theta = 0$

Potential and limitations

- point-like structures ✓
- chatter marks ×

First Approach



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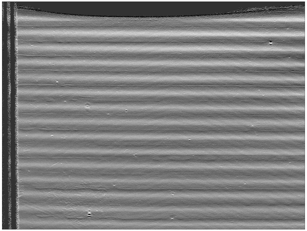
Potential and limitations

- point-like structures ✓
- chatter marks $\times$

Optimal solution

- detect characteristics

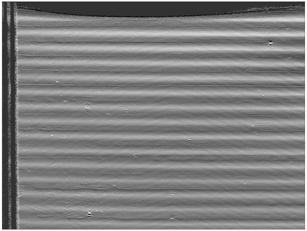
Second Approach



Aim:

- anisotropic scaling parameters $s_1, s_2 \in \mathbb{R} \setminus \{0\}$

Second Approach



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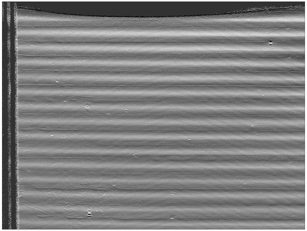

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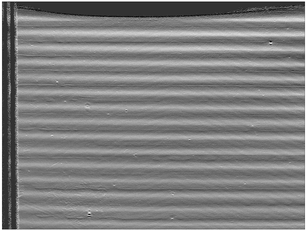

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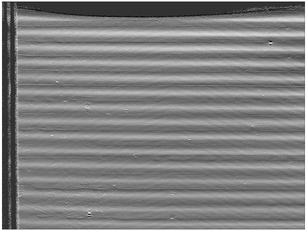
- role model: continuous wavelet transform



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- rotation parameter $\theta \in [0, 2\pi)$
- translation parameter $b \in \mathbb{R}^2$
- role model: continuous wavelet transform
  - independence of dyadic scaling parameter

Second Approach

Considered groups

- $D_{dil} = \left\{ \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} : s_1, s_2 \in \mathbb{R} \setminus \{0\} \right\}$
- $D_{rot} = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in [0, 2\pi) \right\}$
- $D_{tra} = \left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} : b_1, b_2 \in \mathbb{R} \right\}$

Second Approach

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$\Rightarrow$ construction of a wavelet-like transform with these three components

Outline

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1. Representation Theory
2. Rotational Anisotropic Wavelet Transform

Basic Definitions

Definition

A locally compact topological group is a group G with topology such that

- $G \times G \to G, (a, b) \mapsto ab$
- $G \to G, a \mapsto a^{-1}$

are continuous and G is locally compact.

Basic Definitions

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Definition

A locally compact topological group is a group G with topology such that

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are continuous and G is locally compact.

Examples

- $(\mathbb{R}^n, +)$
- every closed subgroup of $Gl_n(\mathbb{R})$ with matrix multiplication

Representation

In the following:

- $H \neq 0$ Hilbert space,
- $U(H)$ unitary operators on H,
- G locally compact topological group

Representation

- $H \neq 0$ Hilbert space,
- $U(H)$ unitary operators on H,
- G locally compact topological group

Definition

A unitary representation is a homomorphism $\pi : G \mapsto U(H)$,

- $\pi(ab) = \pi(a)\pi(b)$
- $\pi(a^{-1}) = \pi(a)^{-1}$

that is (strongly) continuous with respect to the strong operator topology.

- $a \mapsto \pi(a)x$ is continuous from G to H for any $x \in H$

Square Integrable Representation

Definition

Let

- π be a representation of G in H
- μ be a left Haar measure of G

If there exists one vector $0 \neq \varphi \in H$, such that

$$\int_G |\langle \pi(a)\varphi, \varphi\rangle_H|^2 d\mu < \infty$$

then π is square integrable and φ is called admissible for π and μ.

Square Integrable Representation

Theorem[1]

Let π be a unitary irreducible square integrable representation of G in H

[1] Grossmann, Morlet, Paul: Transforms associated to square integrable group representations. I. General results, 1985

Square Integrable Representation

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Theorem[1]

Let π be a unitary irreducible square integrable representation of G in H then

- the set of admissible vectors is dense in H
- the operator $V_\varphi : H \to L_2(G)$, given by

$$V_\varphi f(a) := \langle f, \pi(a)\varphi\rangle_H,$$

  is a multiple of the isometry

[1] Grossmann, Morlet, Paul: Transforms associated to square integrable group representations. I. General results, 1985

Back to the Aim

We have:

- $G_{rot}, G_{dil}, G_{tra}$ are locally compact topological groups ✓

We need:

- suitable representation for a semidirect product

Left Regular Representation

Definition

The left regular representation π_L of G on $L_2(G, d\mu_L)$ is given by

$$(\pi_L(a) f)(x) = f\left(a^{-1}x\right).$$

Representation for Semidirect Products

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Definition

Let

- $G = M \ltimes_\sigma N$ be a semidirect product group with homomorphism σ
- M, N be locally compact groups

The left regular representation π_L of G on $L_2\left(G, d\mu_L\right)$ is given by

$$\left(\pi_L\left(a, b\right) f\right)\left(x, y\right) = f\left(a^{-1}x, \sigma_{a^{-1}}(y \circ b^{-1})\right).$$

Definition

Let

- $G = M \ltimes_\sigma N$ be a semidirect product group
- M, N be locally compact groups
- N be abelian

The left quasiregular representation π_L of G on $L_2(N, d\mu_L)$ is given by

$$(\pi_L(a,b)f)(x) = \delta(a)^{-\frac{1}{2}} f\left(\sigma_{a^{-1}}(y \circ b^{-1})\right).$$

Back to the Aim

We have

- $G_{rot}, G_{dil}, G_{tra}$ are (locally) compact topological groups ✓
- left quasiregular representation π_L of $G = M \ltimes_\sigma N$ on $L_2(N, d\mu_L)$ ✓

We need

- Which groups are admissible?

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Theorem[1]

Let

- M be a subgroup of $Gl_n(\mathbb{R})$
- topological semidirect product $M \ltimes \mathbb{R}^n$
- quasiregular representation has nontrivial subrepresentation with admissible vector

Then M is a closed subgroup of $Gl_n(\mathbb{R})$.

[1] H. Führ, Abstract Harmonic Analysis of Continuous Wavelet Transforms, Springer 2005

Closed Subgroups

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Admissible groups[a,b]

- for $c \in \mathbb{R}$: $D_{she} = \left\{ \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$
- $D_{dil} = \left\{ \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} : s_1, s_2 \in \mathbb{R} \setminus \{0\} \right\}$
- $D_{rot} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : (a, b) \in \mathbb{R}^2 \setminus \{(0,0)\} \right\}$

[a] Bernier and Taylor, Wavelets from square-integrable representations, SIAM, 1996
[b] Führ, Zur Konstruktion von Wavelettransformationen in höheren Dimensionen, 1997

Back to the Aim

We have

- $G_{rot}, G_{dil}, G_{tra}$ are (locally) compact topological groups ✓
- left quasi regular representation π_L of $G = M \ltimes_\sigma N$ on $L_2(N, d\mu_L)$ ✓
- $G_{rot}, G_{dil}, G_{tra}$ are admissible ✓

Components for the New Transform

Requirements

$$D_{rot} = \left\{ R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} : \alpha \in [0, 2\pi) \right\}$$

$$D_{dil} = \left\{ A_s = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} : s_1, s_2 \in \mathbb{R} \setminus \{0\} \right\}$$

$\Rightarrow$ semidirect product $G_{raw} := (D_{rot} \times D_{dil}) \rtimes \mathbb{R}^2$

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$\Rightarrow$ semidirect product $G_{raw} := (D_{rot} \times D_{dil}) \rtimes \mathbb{R}^2$

Problem and solution

- G_{raw}: no group structure
- solution: for fixed α use group structure of D_{dil}

Rotational Anisotropic Wavelet Transform

Definition

Consider

- $\varphi \in L_2\left(\mathbb{R}^2\right)$ admissible for D_{dil}
- $f \in L_2\left(\mathbb{R}^2\right)$
- $s_1, s_2 \in \mathbb{R} \setminus \{0\}, \alpha \in [0, 2\pi)$ and $b \in \mathbb{R}^2$

The rational anisotropic wavelet transform is given by

$$RAW_{\varphi} f\left(s, \alpha, b\right) = \int_{\mathbb{R}^2} f\left(x\right) \overline{\varphi_{s,\alpha,b}\left(x\right)} dx,$$

with

$$\varphi_{s,\alpha,b}\left(x\right) = \left|s_1 s_2\right|^{-\frac{1}{2}} \varphi\left(R_{\alpha} A_s\left(x - b\right)\right).$$

Tensor product wavelets

$$\varphi_1, \varphi_2 \text{ are 1D wavelets} \Rightarrow \int_{\hat{\mathbb{R}}^2} \frac{\left|\hat{\varphi}\left(\xi_1, \xi_2\right)\right|^2}{\left|\xi_1 \xi_2\right|} d\xi_1 d\xi_2 < \infty,$$

where $\varphi\left(x, y\right) = \varphi_1\left(x\right)\varphi_2\left(y\right)$

Frequency Domain

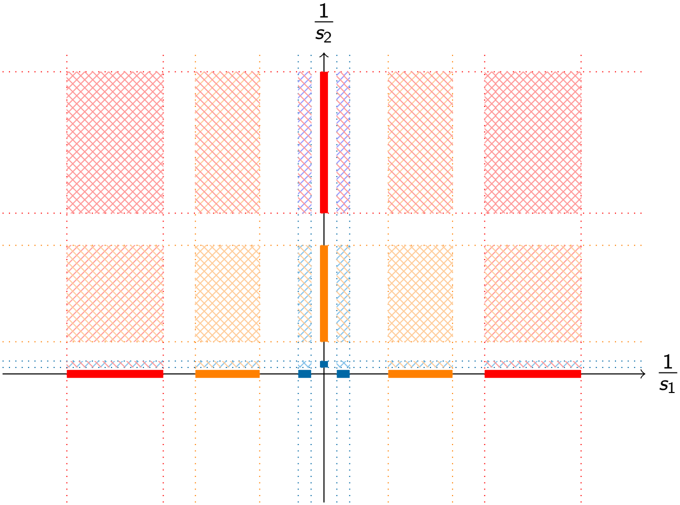


Figure: support of rotational anisotropic wavelets

Essential Support

Lemma [G.]

Let $\varphi_{s,\alpha,b}$ with $s_1, s_2 \in \mathbb{R} \setminus \{0\}$, $\alpha \in [0, 2\pi)$ be a rotational wavelet then

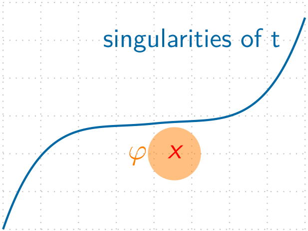
$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^2} \varphi_{s,\alpha,b}(x) = R_\alpha^{-1} A_s^{-1} \operatorname*{ess\,sup}_{x \in \mathbb{R}^2} \varphi(x) \quad \text{and}$$

$$\operatorname*{ess\,sup}_{x \in \hat{\mathbb{R}}^2} \hat{\varphi}_{s,\alpha,b}(\xi) = R_\alpha^{-1} A_s \operatorname*{ess\,sup}_{\xi \in \hat{\mathbb{R}}^2} \hat{\varphi}(\xi).$$

Wavefront Set

In the following:

- tempered distribution $t \in S'(\mathbb{R}^n)$
- x is a regular point
- φ cutoff function



Wavefront Set

tempered distribution $t \in S'(\mathbb{R}^n)$

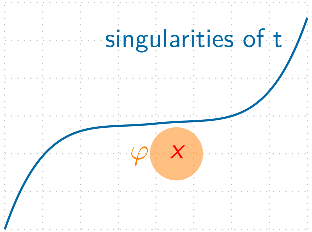
x is a regular point

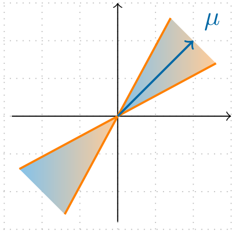
φ cutoff function

Definition

Then a pair $(x, s) \in \mathbb{R}^2 \times \mathbb{R}$ is a regular directed point if there exists a neighbourhood V_s of s such that for all $N \in \mathbb{N}$ and $\mu = (\mu_1, \mu_2)$

$$(\varphi t)^\wedge(\mu) = O\left(\left(1 + |\mu|\right)^{-N}\right) \text{ with } \frac{\mu_2}{\mu_1} \in V_s.$$





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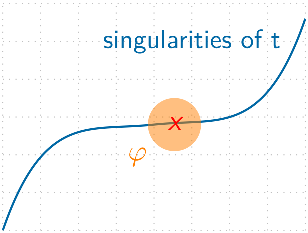
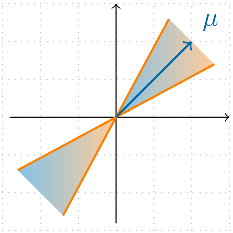




Definition

The wavefront set $WF(t)$ is the complement of the regular directed points.

Wavefront Set

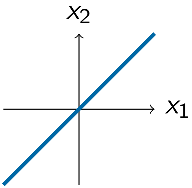


Figure: time domain

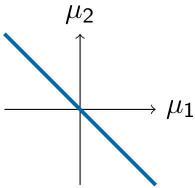


Figure: frequency domain

Wavefront set of line singularity

For $\delta_{x_2=p+qx_1}$ the wavefront set is

$$WF\left(\delta_{x_2=p+qx_1}\right) = \{(x_1, x_2)\,|x_2 = p + qx_1\} \times \left\{-\frac{1}{q}\right\}.$$

Line Singularities

Theorem [G.]

Let $g(x) = \delta_{x_2 = q x_1}(x)$ for $q \neq 0$.
For $b_2 = q b_1$ and $\tan(\alpha) = \frac{1}{q}$

$$RAW_\Psi g(s, \alpha, b) \sim |s_1|^{-\frac{1}{2}} |s_2|^{-\frac{3}{2}}, \quad \text{for } s_1, s_2 \to 0,$$

otherwise $RAW_\Psi v(s, \alpha, b)$ decays rapidly.

Back to the Motivation

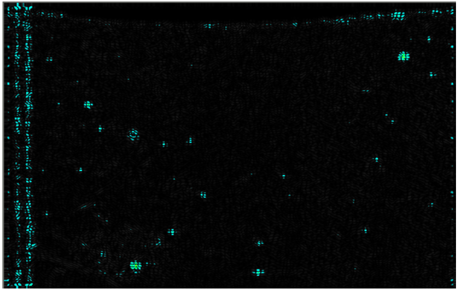


Figure: 2D wavelet transform of product surface with chatter marks

Back to the Motivation

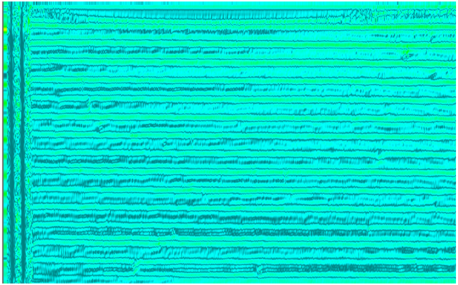
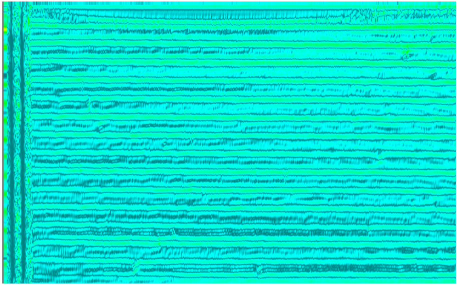


Figure: rotational anisotropic wavelet transform of chatter marks

Back to the Motivation



Improvements

- wavelet like transform with rotation and anisotropic scaling
- detect line singularities (demonstration)
- fast implementation with FFT

Thank you for your attention!