Interpolation with complex B-splines

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The first part of a story in three chapters

- Complex frames and bases for signal / image processing.
  Complex B-splines.
  Interpolation with complex B-spline? For a subset of parameters. (BF)

- Frame construction? Not in the standard way as it is done with B-splines.
  Alternative: Pseudo-splines. (Peter Massopust)

- Technique: Unitary Extension Principle UEP. (Ole Christensen)

- Relation to subdivision.
- Relation to the Dubuc interpolation.

1. Why complex transforms for signal processing?

1. Why not? The Fourier transform is complex-valued, after all.
   Beautiful physical interpretation: Amplitude at a certain frequency.

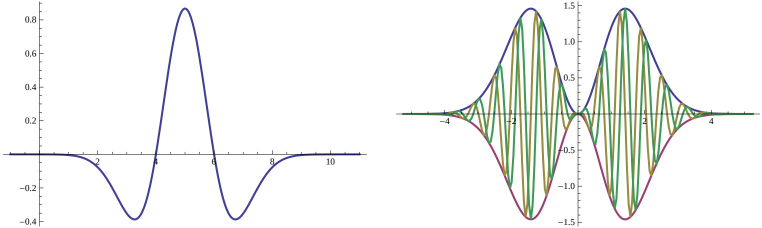
2. Fighting the dogma "Images are real-valued".
   Signals and images are not always real-valued:
   MRI, holography, phase retrieval.

3. Real-valued transforms cannot separate one-sided frequency bands.

4. The phase in images contains edge and detail information.

(BF: Five good reasons for complex-valued transforms in image processing. In: G. Schmeier and A. Zayed (Eds.): New Perspectives on Approximation and Sampling Theory – Festschrift in honor of Paul Butzers 85th birthday. Birkhäuser. 2014.)

Third reason:
Real bases cannot analyze one-sided frequency bands.

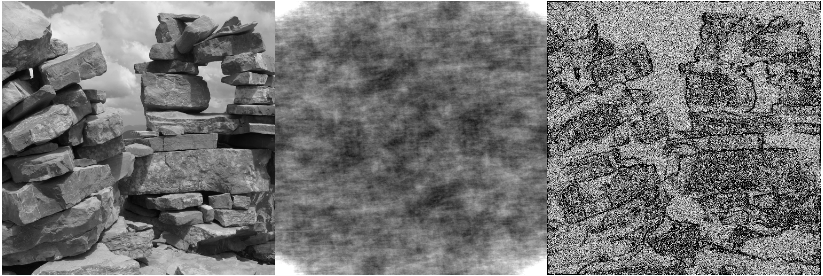


Left: Mexican hat wavelet. Right: Its Fourier transform and spectral envelope.

1. Why complex transforms for signal processing?

Fourth reason: The phase of images contains edge and detail information.

Reconstruction from Fourier amplitude and phase:

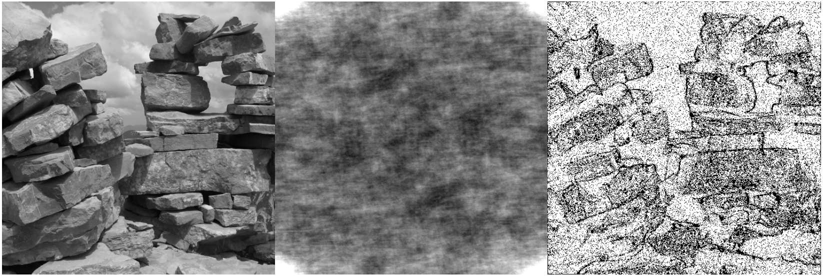


(Cf. A. V. Oppenheim and J. S. Lim, The importance of phase in signals, Proc. IEEE, 1981.)

(Left image: Laurent Condat's Image Database http://www.greyc.ensicaen.fr/∼lcondat/imagebase.html.)

Fourth reason: The phase of images contains edge and detail information.

Reconstruction from Fourier amplitude and phase (denoised):

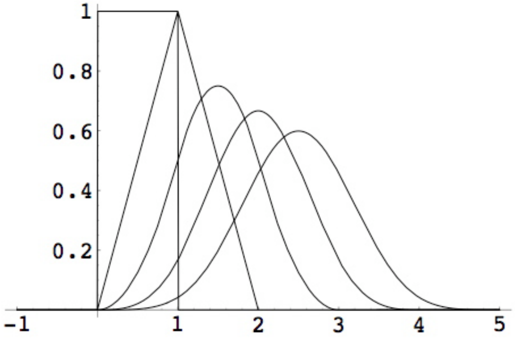


(Cf. A. V. Oppenheim and J. S. Lim, The importance of phase in signals, Proc. IEEE, 1981.)

(Left image: Laurent Condat's Image Database http://www.greyc.ensicaen.fr/∼lcondat/imagebase.html.)

Cardinal B-splines B_n of order n, $n \in \mathbb{N}$, with knots in $\mathbb{Z}$ are defined recursively:

- $B_1 = \chi_{[0,1]}$,
- $B_n = B_{n-1} * B_1$ for $n \geq 2$.



Piecewise polynomials of degree at most n and smoothness $C^{n-1}(\mathbb{R})$.

Classical B-splines

$$B_1 = \chi_{[0,1]}, \quad B_n = B_{n-1} * B_1 \text{ for } n \geq 2,$$

$$\widehat{B}_n(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega} \right)^n$$

- have a discrete indexing.
- have a discrete order of approximation.
- are real-valued.
- have a symmetric spectrum.

Idea: Complex-valued B-splines

defined in Fourier domain:

$$\widehat{B}_z(\omega) = \widehat{B}_{\alpha+i\gamma}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^z,$$

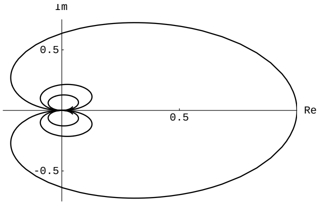
where $z \in \mathbb{C}$, with parameters $\operatorname{Re} z > \frac{1}{2}$, $\operatorname{Im} z \in \mathbb{R}$.

(BF, T. Blu, M. Unser. Complex B-splines. ACHA, 2006.)

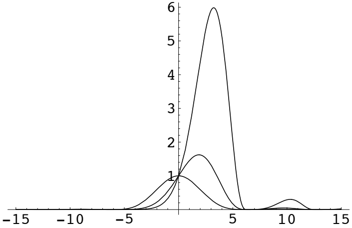
Why is this definition reasonable and useful?

- $\widehat{B}_z(\omega) = \widehat{B}_{\alpha+i\gamma}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^z = \Omega(\omega)^z$ is well defined, since $\Omega(\mathbb{R}) \cap \mathbb{R}_- = \emptyset$.

- $|\widehat{B}_z(\omega)| = |\Omega(\omega)^z| = |\Omega(\omega)|^{\operatorname{Re} z} e^{-\operatorname{Im} z \arg(\Omega(\omega))}$.



$\Omega : \mathbb{R} \to \mathbb{C}$



$|\widehat{B}_{3+i\gamma}|$ for $\gamma = 0, 1, 2$.

- Approximately single-sided frequency analysis!

What do they look like? Representation in time-domain

$$B_z(t) = \frac{1}{\Gamma(z)} \sum_{k \geq 0} (-1)^k \binom{z}{k} (t-k)_+^{z-1}$$

pointwise for all $t \in \mathbb{R}$ and in the $L^2(\mathbb{R})$ norm.

Compare:
The cardinal B-spline B_n, $n \in \mathbb{N}$, has the representation

$$\begin{aligned} B_n(t) &= \frac{1}{(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (t-k)_+^{n-1} \\ &= \frac{1}{\Gamma(n)} \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (t-k)_+^{n-1}. \end{aligned}$$

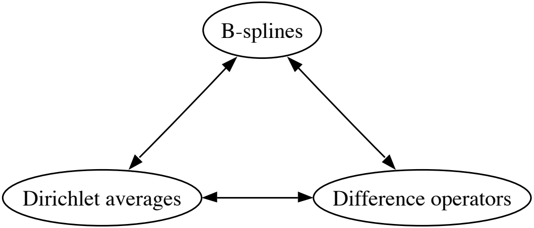
2. Complex B-splines

- Complex B-splines B_z are piecewise polynomials of complex degree.
- Smoothness and decay tunable via the parameter $\operatorname{Re} z$.
  - $B_z \in W_2^r(\mathbb{R})$ for $r < \operatorname{Re} z - \frac{1}{2}$
  - $B_z(x) = \mathcal{O}(x^{-m})$ for $m < \operatorname{Re} z + 1$, $|x| \to \infty$.
- Recursion: $B_{z_1} * B_{z_2} = B_{z_1+z_2}$
- They are scaling functions and generate **multiresolution analyses** and wavelets.
- Simple implementation in Fourier domain $\to$ **Fast algorithm.**
- Nearly optimal **time frequency localization.**
- However, no compact support.

(Here $\operatorname{Re} z, \operatorname{Re} z_1, \operatorname{Re} z_2 > 1$.)

2. Complex B-splines

- Relate fractional or complex **differential operators** with **difference operators.**



Difference operator of complex order:

$$\nabla^z g(t) := \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} g(t-k), \quad z \in \mathbb{C},\ \operatorname{Re} z \geq 1.$$

(P.L. Butzer, M. Hauss, M. Schmidt, Factorial functions and Stirling numbers of fractional orders, Results Math. 1989)

Consequence for complex B-splines:

$$B_z(t) = \frac{1}{\Gamma(z)} \nabla^z t_+^{z-1}.$$

Compare with the cardinal splines

$$B_n(t) = \frac{1}{(n-1)!} \nabla^n t_+^{n-1}.$$

Complex divided differences for the knot sequence $\mathbb{N}_0$:

$$[z; \mathbb{N}_0]g := \sum_{k \geq 0} (-1)^k \frac{g(k)}{\Gamma(z-k+1)\Gamma(k+1)}.$$

Representation for complex B-splines:

$$B_z(t) = z[z, \mathbb{N}_0](t - \bullet)_+^{z-1}$$

(BF, P. Massopust: Splines of Complex Order: Fourier, Filters and Fractional Derivatives. Sampling Theory in Signal and Image Processing, 2011.

BF, P. Massopust: Statistical Encounters with Complex B-Splines. Constructive Approximation, 2009

BF, P. Massopust: Some Remarks about the Connection between Fractional Divided Differences, Fractional B-Splines, and the Hermite-Genocchi Formula. Int. J. of Wavelets, Multiresolution and Information Processing, 2008.)

Theorem: Relations to differential operators
(BF, P. Massopust, 2008)
Let Re $z > 0$ and $g \in \mathcal{S}(\mathbb{R}^+)$. Then

$$[z; \mathbb{N}_0]g = \frac{1}{\Gamma(z+1)} \int_{\mathbb{R}} B_z(t) g^{(z)}(t)\, dt,$$

where $g^{(z)} = W^z g$ is the complex Weyl derivative.

For $n = \lceil \mathrm{Re}\, z \rceil$, $\nu = n - z$,

$$W^z g(t) = (-1)^n \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(\nu)} \int_t^{\infty} (x-t)^{\nu-1} g(x)\, dx \right].$$

Sketch of proof:

$$\begin{aligned} \frac{1}{\Gamma(z+1)} \int_{\mathbb{R}} B_z(t) g^{(z)}(t)\, dt &= \frac{1}{\Gamma(z+1)} \int_{\mathbb{R}} z[z, \mathbb{N}_0](t - \bullet)_+^{z-1} W^z g(t)\, dt \\ = [z, \mathbb{N}_0] \frac{1}{\Gamma(z)} \int_{\bullet}^{\infty} (t-\bullet)_+^{z-1} W^z g(t)\, dt &= [z, \mathbb{N}_0] W^{-z} W^z g = [z, \mathbb{N}_0] g. \end{aligned}$$

Theorem: Generalized Hermite-Genocchi-Formula
Let Δ^∞ be the infinite-dimensional simplex

$$\Delta^\infty := \{u := (u_j) \in (\mathbb{R}_0^+)^{\mathbb{N}_0} | \sum_{j=0}^{\infty} u_j = 1\} = \varprojlim \Delta^n,$$

μ_e^∞ the generalized Dirichlet measure $\mu_e^\infty = \varprojlim \Gamma(n+1)\lambda^n$, and λ^n the Lebesgue measure on Δ^n. Then

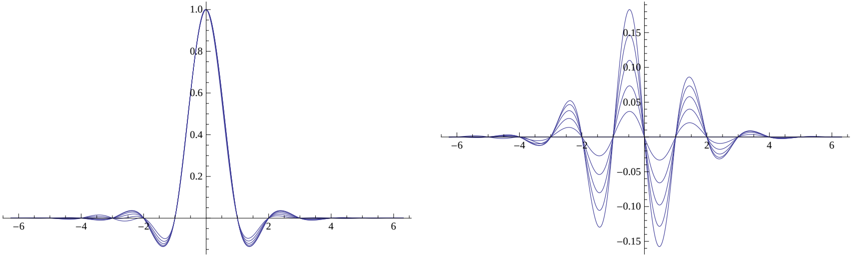
$$\begin{aligned} [z, \mathbb{N}_0]g &= \frac{1}{\Gamma(z+1)} \int_{\Delta^\infty} g^{(z)}(\mathbb{N}_0 \cdot u) d\mu_e^\infty(u) \\ &= \frac{1}{\Gamma(z+1)} \int_{\mathbb{R}} B_z(t) g^{(z)}(t)\, dt \end{aligned}$$

for all real-analytic $g \in \mathcal{S}^\omega(\mathbb{R}^+)$.

How about interpolation with complex B-splines?

How about interpolation with complex B-splines?

Some interpolating complex splines for $\operatorname{Re} z = 4$, $\operatorname{Im} z = 0, 0.1, \ldots, 0.5$: (Real and imaginary part)



(BF, Peter Massopust, Ramūnas Garunkštis, and Jörn Steuding. Complex B-splines and Hurwitz zeta functions. LMS Journal of Computation and Mathematics, 16, pp. 61–77, 2013.)

3. Interpolation problem for complex B-splines

Aim: Interpolating splines L_z of fractional or complex order.

Interpolation problem:

$$L_z(m) = \sum_{k\in\mathbb{Z}} c_k^{(z)} B_z\,(m-k) = \delta_{m,0}, \quad m \in \mathbb{Z},$$

Fourier transform:

$$\widehat{L}_z(\omega) = \frac{\widehat{B}_z(\omega)}{\sum_{k\in\mathbb{Z}} \widehat{B}_z(\omega + 2\pi k)} = \frac{1/\omega^z}{\sum_{k\in\mathbb{Z}} \frac{1}{(\omega+2\pi k)^z}}, \quad \mathrm{Re}\, z \geq 1.$$

Question: When is the denominator well-defined?

Rescale denominator.

$$\widehat{L}_z(\omega) = \frac{1/\omega^z}{\displaystyle\sum_{k\in\mathbb{Z}} \frac{1}{(\omega+2\pi k)^z}}, \quad \operatorname{Re} z \geq 1$$

Let $\omega \neq 0$ and $a := \omega/2\pi$.

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \frac{1}{(k+a)^z} &= \sum_{k=0}^{\infty} \frac{1}{(k+a)^z} + e^{\pm i\pi z} \sum_{k=0}^{\infty} \frac{1}{(k+1-a)^z} \\ &= \boxed{\zeta(z,a) + e^{\pm i\pi z}\,\zeta(z,1-a).} \end{aligned}$$

This is a sum of Hurwitz zeta functions $\zeta(z,a)$, $0 < a < 1$.

4. Special case: Fractional order

Theorem: (Spira, 1976)

If Re $z \geq 1 + a$, then $\zeta(z, a) \neq 0$.

Thus: For $0 < a < 1$ and for $\sigma \in \mathbb{R}$ with $\sigma \geq 2$ and $\sigma \notin 2\mathbb{N} + 1$:

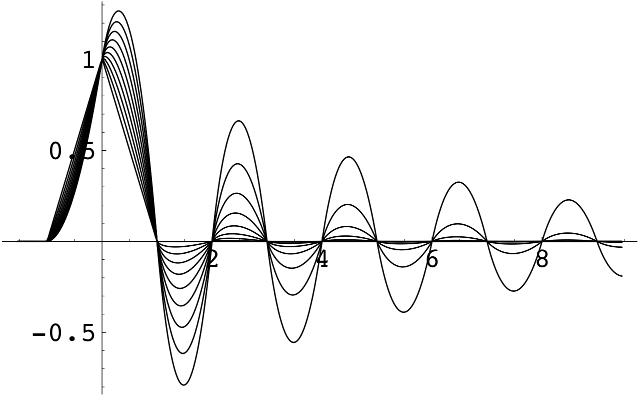
$$\boxed{\zeta(\sigma, a) + e^{\pm i\pi\sigma}\,\zeta(\sigma, 1 - a) \neq 0}.$$

Theorem: (BF, P. Massopust 2011)

For $\sigma \in \mathbb{R}$ with $\sigma \geq 2$ and $\sigma \notin 2\mathbb{N} + 1$ the complex spline L_z solves the interpolation problem.

Remember: Interpolation with B-splines of odd degree is covered by Schoenberg's symmetric versions of the splines.

4. Special case: Fractional order



Interpolating spline L_σ in time domain for $\sigma = 2.0, 2.1, \ldots, 2.9$.
$\sigma = 2$: Classical case of linear interpolation.

5. Interpolation with complex B-splines

Question:
Are there complex exponents $z \in \mathbb{C}$ with $\operatorname{Re} z > 3$ and $\operatorname{Re} z \notin 2\mathbb{N} + 1$, such that the denominator of the Fourier representation of the fundamental spline satisfies

$$\zeta(z, a) + e^{\pm i\pi z}\zeta(z, 1 - a) \neq 0$$

for all $0 < a < 1$.

Consequence:
The interpolation construction also holds for certain B-splines L_z of complex order z.

Why was the problem on zero-free regions so resisting?

$$f_{\pm}(z,a) := \zeta(z,a) + e^{\pm i\pi z}\zeta(z,1-a)$$

Closely related to the famous Riemann hypothesis.

(Bernhard Riemann, Source: Wikipedia)

5. Interpolation with complex B-splines

Why was the problem on zero-free regions so resisting?

$$f_{\pm}(z,a) := \zeta(z,a) + e^{\pm i\pi z}\zeta(z,1-a)$$

- Involves the Hurwitz zeta function aside the “interesting domain” around $\operatorname{Re} z = \frac{1}{2}$.
- Rouché’s theorem seems to be suitable to find zeros, but not to exclude them.
  The problem is that the estimates have to be uniform in a.
- Numerical examples fail, because the Hurwitz zeta function is implemented via meromorphic approximations.

However, there exist classes of interpolating B-splines of complex order.

Curves of zeros:

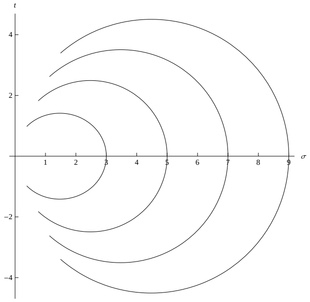
Denote

$$f_{\pm}(z,a) := \zeta(z,a) + e^{\pm i\pi z}\zeta(z,1-a).$$

Consider the zero trajectories

$$\frac{\partial z(a)}{\partial a} = -\frac{\frac{\partial f_{\pm}(z,a)}{\partial a}}{\frac{\partial f_{\pm}(z,a)}{\partial z}},$$

where $z = z(a)$ and $f_{\pm}(z(a),a) = 0$ with initial conditions provided at the zeros $z = 2n+1$, $n \in \mathbb{N}$, $a = 1/2$.



$0.001 < a < 0.999$

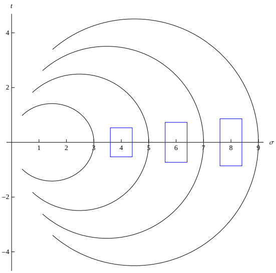
5. Interpolation with complex B-splines

Functional equation:

$$f_\pm(z, 1-a) = e^{\pm i\pi z} f_\mp(z, a).$$

We can reduce the problem to $0 < a \leq \frac{1}{2}$.
Define

$$g_\pm(z, a) = a^z f_\pm(z, a) = 1 + e^{\pm i\pi z}\left(\frac{a}{1-a}\right)^z + B$$



Basic estimations of the first two terms and B yield squares of zero-free regions:

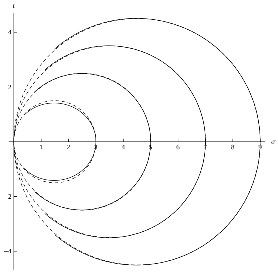
$$S = \{z \in \mathbb{C} : z = 2(n+1) + s,\ |\mathrm{Re}\, s| \leq X < \frac{1}{2},\ |\mathrm{Im}\, s| \leq Y\}.$$

Consider again the first terms:

$$1 + e^{\pm i\pi z}\left(\frac{\alpha}{1-\alpha}\right)^{z} = 0 \quad \text{for } 0 < \alpha < 1,$$

describes circles:

$$z \in \left\{ s : |s - k + \frac{1}{2}| = |k + \frac{1}{2}|, k \in \mathbb{Z} \right\}.$$



Crescent shaped zero-free regions
For

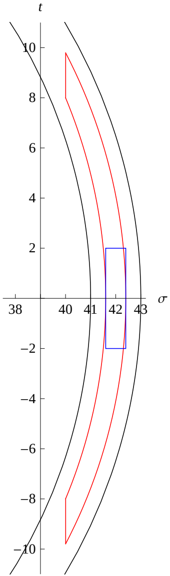
$$\left\{ s : \left| s - \frac{2k+2+\delta}{2} \right| = \frac{2k+2+\delta}{2},\ 0 < |\delta| < \delta_0, \right.$$

$$\left. \sigma_0 \leq \mathrm{Re}\, s \leq 2n+2+\delta_0 \right\}$$

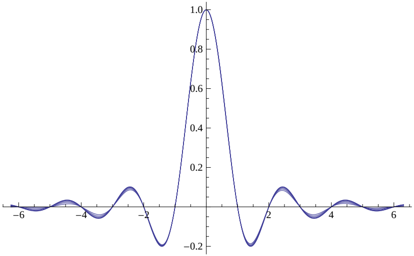
with $\delta_0 < 1$ and $2 < \sigma_0 < 2n+2+\delta_0$, the first two terms are located between the circles of zeros,

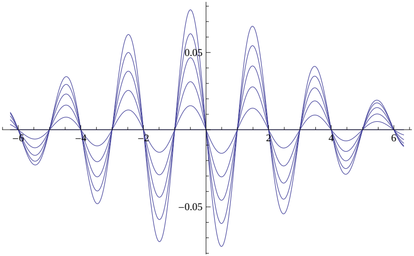
- $\left| 1 + e^{\pm i\pi s} \left(\frac{\alpha}{1-\alpha} \right)^s \right| \geq \frac{1}{2}$,
- and the term B satisfies $|B| < \frac{1}{2}$.

$\Rightarrow$ **Zero Free Regions.**



Example:





Interpolating complex B-splines with $\operatorname{Re} z = 10$, $\operatorname{Im} z = 0, 0.1, \ldots, 0.5$ (Real and imaginary part).

Summary

- Complex B-splines are a natural extension of the Schoenberg splines to a complex degree.
- They are related to fractional difference operators, fractional differential operators, Dirichlet means, ...
- The allow for interpolation as their real valued relatives, if their complex degree z stays away from the zero curves of a certain sum of Hurwitz zeta functions.

Thank you!