Detection of Hidden Frequencies: An OPUC-based Approach

Frank Filbir

Technische Universität München
Helmholtz Center Munich

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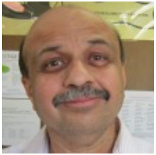
In collaboration with



Hrushikesh N. Mhaskar

Jürgen Prestin

Outline

(1) **The Frequency Detection Problem**

(2) **Classical Method: de Prony's Legacy**

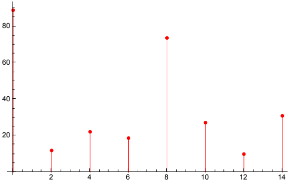
(3) **OPUC I: Classical Approach**

(4) **OPUC II: Kernel-based Approach (Noiseless Recovery)**

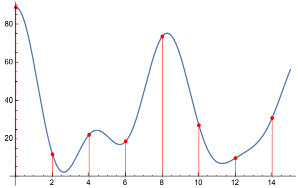
(5) **OPUC III: Kernel-based Approach (Recovery from Noisy Data)**

Super-Resolution

➤ Blurring



High-resolution signal $\hat{x}$



Blurred signal $\hat{y} = \hat{x} * h$

In frequency domain

$$y(t) = x(t)\, h(t),$$

where $\hat{h}$ is a low-pass filter, i.e. $\mathrm{supp}(h) = [-T, T]$.

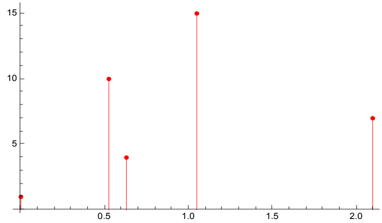
➤ Super-resolution

Recover $\hat{x}$ from low-frequency information samples $y(k),\ k = 0, 1, \ldots .$

Ill-posed extrapolation problem !

The Frequency Detection Problem

➤ **Set up**



Consider spike train

$$\hat{x} = \sum_{j=-I}^{I} a_j \, \delta_{\omega_j}$$

with

$$a_j \in \mathbb{C}, \; a_{-j} = \overline{a_j},$$

$$\omega_j \in \mathbb{R}, \quad \omega_{-j} = -\omega_j, \quad \text{and}$$

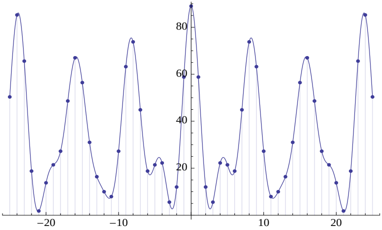
$$0 = \omega_0 < \omega_1 < \cdots < \omega_I < \pi.$$

Fourier Stieltjes transform of measure $\hat{x}$

$$x(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\omega} \, \mathrm{d}\hat{x}(\omega) = \sum_{j=-I}^{I} a_j \, \mathrm{e}^{\mathrm{i}\omega_j t}$$

The Frequency Detection Problem

➤ Data



Sampling of signal

$$x(t) = \sum_{j=-I}^{I} a_j \, e^{i\, t\, \omega_j},$$

at finitely many integer nodes

$$x(k) = \sum_{j=-I}^{I} a_j \, e^{i\, k\, \omega_j}, \quad k = 0, \ldots, K.$$

➤ Inverse problem

Determine frequencies ω_j, and a_j, $j = -I, \ldots, I$ from data

$$\{x(0), \ldots, x(K)\}.$$

The difficult part is the determination of the ω_j's !

The Frequency Detection Problem

➤ **Inverse problem**

Inversion of the non-linear mapping

$$\mathcal{P} : \mathbb{C}^{2(I+1)} \to \mathbb{C}^K, \qquad \mathcal{P}(a_0, \ldots, a_I; \omega_0, \ldots, \omega_I) = \big(x(0), \ldots, x(K)\big)$$

➤ **Different Methods**

Prony's method

ESPRIT & MUSIC [R. Roy, T. Kailath ...]

Least squares [D. Batenkov, Y. Yodom...]

Matrix pencil methods [G. Beylkin, L. Monzón 2005, D. Potts, M. Tasche, G. Plonka, T. Peter...]

TV minimization [E. Candés, C. Fernandes-Granda, L. Demanet, N. Nguyen ...]

OPUC [W.B. Jones, E. Saff, O. Njåstad, H.N. Mhaskar, J. Prestin, F.F.]

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➤ De Prony's idea

Gaspard Clair François Marie Riche de Prony

1755-1839

Relate to the signal

$$x(t) = \sum_{j=-I}^{I} a_j \, e^{i\,\omega_j\,t}$$

a polynomial

$$P(z) = \prod_{j=-I}^{I} (z - \zeta_j) = \sum_{k=0}^{2I+1} c_k \, z^k, \qquad \zeta_j = e^{i\omega_j}$$

For $m = 0, 1 \ldots$ we have

$$\begin{aligned} \sum_{k=0}^{2I+1} c_k \, x(k+m) &= \sum_{k=0}^{2I+1} c_k \sum_{j=-I}^{I} a_j \zeta_j^{k+m} \\ &= \sum_{j=-I}^{I} a_j \, \zeta_j^m \, P(\zeta_j) \\ &= 0 \end{aligned}$$

➤ De Prony's idea

Compute coefficients in

$$P(z) = \sum_{k=0}^{2I+1} c_k \, z^k$$

by linear system

$$\begin{pmatrix} x(0) & x(1) & \dots & x(2I+1) \\ x(1) & x(2) & \dots & x(2I+2) \\ \vdots & \vdots & & \vdots \\ x(2I+1) & x(2I+2) & \dots & x(2(2I+1)) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2I+1} \end{pmatrix} = \mathbf{0}.$$

Assumption

Number of samples

$$\sharp\{x(0), \dots, x(2N)\} \geq 2(2I+1), \quad \text{i.e.} \quad 2I+1 \leq N.$$

➤ Computation of frequencies

Compute zeros of

$$P(z) = \sum_{k=0}^{2I+1} c_k \, z^k$$

by solving eigenvalue problem

$$\boldsymbol{C}\,\boldsymbol{v} = \lambda\,\boldsymbol{v},$$

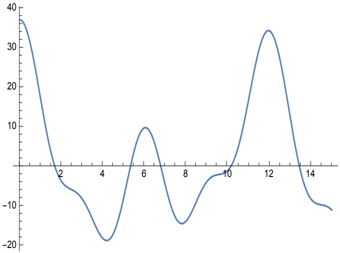
where

$$\boldsymbol{C} = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_{2I} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 1 & -c_{2I+1} \end{pmatrix}$$

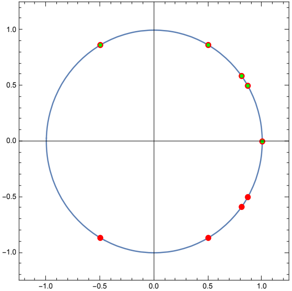
is the companion matrix.

➤ Example

$$x(t) = 1 + 10\cos\left(\frac{\pi}{6}t\right) + 4\cos\left(\frac{\pi}{5}t\right) + 15\cos\left(\frac{\pi}{3}t\right) + 7\cos\left(\frac{2\pi}{3}t\right)$$



Signal $x(t)$



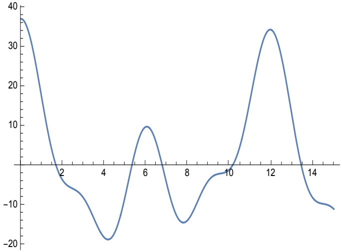
Frequency locations

+ Method uses minimal number of samples.
- Method is sensitive to noise and spacing of the ω_j's.

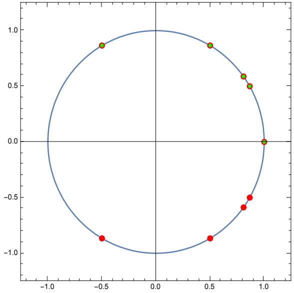
De Prony's Legacy

➤ Example

$$x(t) = 1 + 10\cos\left(\frac{\pi}{6}t\right) + 4\cos\left(\frac{\pi}{5}t\right) + 15\cos\left(\frac{\pi}{3}t\right) + 7\cos\left(\frac{2\pi}{3}t\right)$$



Signal $x(t)$



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OPUC I: Classical Approach

➤ Reminder

$$x(t) = \sum_{j=-I}^{I} a_j \, \mathrm{e}^{\mathrm{i}\,\omega_j\,t},$$

with $a_{-j} = \overline{a_j}$, $\omega_{-j} = -\omega_j$, and $0 = \omega_0 < \omega_1 < \cdots < \omega_I < \pi$.

Assume $I > 0$ is known (or predetermined by some model selection method).

➤ Moments

Let $N \in \mathbb{N}$. Consider samples

$$x_N(k) = \begin{cases} x(k), & 0 \leq k \leq 2N, \\ 0, & \text{otherwise.} \end{cases}$$

and corresponding **autocorrelation sequence**

$$\mu_k^{(N)} = \sum_{m=0}^{2N} x_N(m)\, x_N(m+k).$$

$(\mu_k^{(N)})_{k\in\mathbb{Z}}$ is a positive definite sequence, i.e.

$$\sum_{k=0}^{n} \sum_{\ell=0}^{n} c_k\, \overline{c_\ell}\, \mu_{k-\ell}^{(N)} \geq 0.$$

➤ Representation

Bochner's Theorem: There is a positive measure μ_N on $\mathbb{T}$, s.t.

$$\mu_k^{(N)} = \int_{-\pi}^{\pi} \mathrm{e}^{ikt}\, \mathrm{d}\mu_N(t).$$

The measure is given by Z-transform of sequence $\{x_N(k)\}$, i.e.

$$X_N(z) = \sum_{m=0}^{2N} x_N(m)\, z^{-m}.$$

We have

$$\int_{-\pi}^{\pi} \mathrm{e}^{ikt} \underbrace{\frac{1}{2\pi} \left| X_N(\mathrm{e}^{it}) \right|^2 \mathrm{d}t}_{\mathrm{d}\mu_N(t)} = \sum_{m=0}^{2N} x_N(m)\, x_N(m+k) = \mu_k^{(N)}$$

Theorem [Jones, Njåstad, Saff, '90]

We have

$$\frac{1}{2N}\, \mu_N \to \sum_{j=-I}^{I} |a_j|^2\, \delta_{\zeta_j}, \quad \zeta_j = \mathrm{e}^{i\omega_j},$$

for $N \to \infty$ w.r.t. weak-$*$ topology on $M_b(\mathbb{T})$.

Why is this result of any help ?

➤ Basics on OPUC

Let μ be a probability measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Define the related moment sequence $(\mu_n)_{n\in\mathbb{Z}}$ by

$$\mu_n = \int_{\mathbb{T}} z^n \, \mathrm{d}\mu(z), \quad n \in \mathbb{Z}.$$

Note that

$$\sum_{n=0}^{N} \sum_{m=0}^{N} c_n \overline{c_m} \, \mu_{n-m} = \int_{\mathbb{T}} \Big| \sum_{n=0}^{N} c_n \, z^n \Big|^2 \mathrm{d}\mu(z) \geq 0,$$

i.e. $(\mu_n)_n$ is a **positive definite sequence**.

The Töplitz determinant is defined as

$$D_n = \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_0 \end{vmatrix} \qquad D_{-1} := 1.$$

We have $D_n > 0$ for all $n = 0, 1, \ldots$.

➤ Basics on OPUC

Related to the moment sequence $(\mu_n)_n$ resp. the measure μ there is a system of **monic orthogonal polynomials** $\{\phi_n\}_{n\in\mathbb{N}_0}$,

$$\phi_n(z) = z^n + \ldots, \qquad \int_{\mathbb{T}} \phi_n(z)\,\phi_m(z)\,\mathrm{d}\mu(z) = \delta_{n,m}\,\frac{D_n}{D_{n-1}}$$

which are given as

$$\phi_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_{-1} & \mu_{-2} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \mu_{-1} & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \cdots & \mu_{-1} \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix}.$$

This can be seen from the orthogonality relation

$$\int_{\mathbb{T}} z^m\,\phi_n(z)\,\mathrm{d}\mu(z) = \delta_{n,m}\,\frac{D_n}{D_{n-1}}.$$

The ϕ_n are also called **Szegö polynomials**.

➤ Szegö recurrence relation

Let $\{\phi_n\}_{n\in\mathbb{N}_0}$, be a system of OPUC. Then

$$\phi_{n+1}(z) = z\,\phi_n(z) - \alpha_n\,\phi_n^*(z), \quad \phi_0(z) = 1,$$

where

- $\phi_n^*(z) = z^n\overline{\phi_n(1/\bar{z})}$ is the so-called reciprocal polynomial,
- α_n is the Verblunsky coefficient, where

$$\alpha_n = \overline{\phi_{n+1}(0)} = \frac{1}{D_n}\begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n-1} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+2} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_{-1} \\ 1 & 0 & \cdots & 0 \end{vmatrix} = \frac{1}{D_n}\begin{vmatrix} \mu_{-1} & \cdots & \mu_{-n-1} \\ \mu_0 & \cdots & \mu_{-n+2} \\ \vdots & \vdots & \vdots \\ \mu_{n-1} & \cdots & \mu_{-1} \end{vmatrix}$$

➤ Zeros

All zeros of ϕ_n lie in

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

➤ Zeros

Moreover,

$$\phi_n(z) = \det(z\boldsymbol{I} - \boldsymbol{C}^{(n)})$$

where $\boldsymbol{C}^{(n)}$ is the n-th cut-off of the so-called CMV[1] matrix

$$\boldsymbol{C} = \begin{pmatrix} * & * & * & & & & \\ * & * & * & * & & & \\ * & * & * & * & * & & \\ & * & * & * & * & * & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where the entries are given in terms of the Verblunsky coefficients

$$C_{k,k+j} = f(\{\alpha_n\}) = f(\{\mu_n\}), \quad j = 0, \pm 1, \pm 2.$$

The **zeros** of ϕ_n are given as **eigenvalues** of $\boldsymbol{C}^{(n)}$

[1] named after Cantero, Moral, Velazquez

OPUC I: Classical Approach

➤ Measures

We have two measures on $\mathbb{T}$

$$\begin{aligned} d\mu_N(t) &= \frac{1}{2\pi}\left|\sum_{m=0}^{2N} x_N(m)\, e^{-imt}\right|^2 dt, \\ \mu &= \sum_{j=-I}^{I} |a_j|^2\, \delta_{\zeta_j}, \quad \zeta_j = e^{i\omega_j}. \end{aligned}$$

Accordingly we have the two systems of monic OPUC

$\{\phi_{N,n}\}_{n\in\mathbb{N}_0}$ OPUC w.r.t. μ_N,

$\{\phi_n\}_{n\in\mathbb{N}_0}$ OPUC w.r.t. μ.

➤ Reminder

We have

$$\frac{1}{2N}\,\mu_N \to \sum_{j=-I}^{I} |a_j|^2\, \delta_{\zeta_j}, \quad \zeta_j = e^{i\omega_j},$$

in weak-$*$ topology as $N \to \infty$.

Why is this result of any help?

➤ Finite system of OPUC

Since μ has finite support we have

$$D_n > 0 \quad \text{for } 0 \leq n \leq 2I, \qquad D_{2I+1} = 0,$$

where $D_n = \det\left[(\mu_{k-\ell})_{k,\ell=0}^{n}\right]$ and μ_k are the moments w.r.t. μ .

Hence the OPUC system $\{\phi_n\}$ w.r.t. μ is finite.

Moreover,

$$\phi_{2I+1}(z) = \prod_{j=-I}^{I} (z - \zeta_j), \quad \zeta_j = \mathrm{e}^{\mathrm{i}\omega_j}.$$

Proposition [Jones, Njåstad, Saff, 1990]

For $k \in \mathbb{Z}$ *fixed we have*

$$\frac{1}{2N}\mu_k^{(N)} = \mu_k + \mathcal{O}(1/N).$$

Theorem [Jones, Njåstad, Saff, 1990]

Assume $a_0 > 0$. *Then for each* $1 \leq n \leq 2I+1$ *we have*

$$\lim_{N\to\infty} \phi_{N,n}(z) = \phi_n(z), \quad z \in \mathbb{C}.$$

In particular

$$\lim_{N\to\infty} \phi_{N,2I+1}(z) = \phi_{2I+1}(z) = \prod_{j=-I}^{I} (z - \zeta_j), \quad z \in \mathbb{C}.$$

On every compact set $K \subset \mathbb{C}$ *we have*

$$|\phi_{N,n}(z) - \phi_n(z)| \leq C_K\, N^{-1}, \quad z \in K, \quad N \in \mathbb{N}.$$

Corrolary [Jones, Njåstad, Saff, 1990]

Assume $a_0 > 0,\ 1 \leq n \leq 2I+1$. *Let* $z_{2I+1,j}^{(N)}$ *be the zero of* $\phi_{N,2I+1}$ *which is closest to* $\zeta_j = \mathrm{e}^{\mathrm{i}\omega_j}$. *Then for* $j = -I, \ldots, I$ *we have*

$$|z_{2I+1,j}^{(N)} - \zeta_j| \leq C\, N^{-1}.$$

➤ **Wiener-Levinson algorithm**

Compute zeros $z_{2I+1,j}^{(N)}, j = 0, \ldots, 2I+1$, of $\phi_{N,2I+1}$ via

$$\boldsymbol{C}^{(2I+1)} \boldsymbol{v} = z\, \boldsymbol{v},$$

where the matrix is the five diagonal CMV matrix

$$\boldsymbol{C}^{(2I+1)} = \begin{pmatrix} * & * & * & & & & \\ * & * & * & * & & & \\ * & * & * & * & * & & \\ & * & * & * & * & * & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

with entries $\boldsymbol{C}_{k,k+j}^{(2I+1)}$ computable from moments μ_n.

This procedure is known as the **Wiener-Levinson algorithm**.

OPUC I: Classical Approach

➤ Pros and Cons

+ Spectral theory of the CMV matrix well studied.

  [B. Simon, *Orthogonal Polynomials on the Unit Circle*, AMS, Vol 54, I & II]

- Convergence of the Wiener-Levinson algorithm is slow ($\approx N^{-1}$).

  Improvements

  Consider modified moments

$$\mu_k^{(N,p)} = \sum_{m=0}^{pN} w_{N,k}^{(p)}\, x(m)\, x(m+k),$$

where $w_{N,k}^{(p)}$ are defined by

$$(1+t+\cdots+t^N)^p = \sum_{k=0}^{pN} w_{N,k}^{(p)}\, t^k.$$

  + Convergence rate $\approx N^{-p}$.
  - Loss of OPUC structure.

[K. C. Pan, *A refined Wiener-Levinson method in frequency analysis*, SIAM J. Math. Anal. 1996]

Can we have both, fast convergence and OPUC structure?

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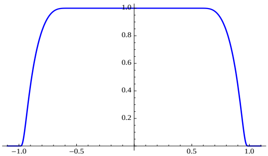
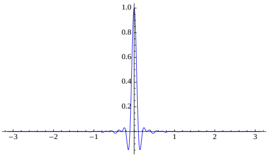
OPUC II: Kernel-based Approach (Noiseless Recovery)

➤ Localized kernel



Let $H : \mathbb{R} \to \mathbb{R}$ be a even **smooth filter function** with compact support, i.e.

$$H \in C^s(\mathbb{R}), \qquad \mathrm{supp}(H) = [-1, 1].$$



For $N \geq 1$ define a kernel by

$$\Phi_N(t) = \sum_{m \in \mathbb{Z}} H\left(\frac{m}{N}\right) \mathrm{e}^{imt}.$$

Remark:

Φ_N is a trigonometric polynomial of degree $N - 1$.

OPUC II: Kernel-based Approach (Noiseless Recovery)

Theorem [Mhaskar, Prestin, F.]

Let $N \geq 1$ *be an integer and* $H : \mathbb{R} \to [0,1], H \in C^s(\mathbb{R})$ *be a filter function with* $H(t) = 0$ *for* $|t| \geq 1$ *. If*

$$N \geq \left(\frac{\|H^{(2)}\|_\infty}{3\,\|H\|_1} \right)^{1/2},$$

then

$$\frac{N\,\|H\|_1}{2} \leq \max_{t \in \mathbb{R}} \left| \Phi_N(t) \right| = \Phi_N(0) \leq 2N - 1.$$

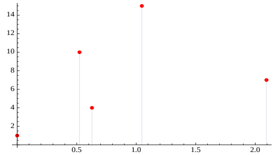
Moreover,

$$\left| \Phi_N(t) \right| \leq C\, \frac{\Phi_N(0)}{(N\,|t|\,)^s}.$$

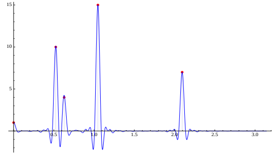
Hence Φ_N decays away from $t = 0$ like $|t|^{-s}$.

OPUC II: Kernel-based Approach (Noiseless Recovery)

➤ Using the kernel



Discrete measure



Kernel approximation to discrete measure

Discrete measure

$$\mu = \sum_{j=-I}^{I} |a_j|^2 \, \delta_{\zeta_j}$$

generated a finite system of OPUC

$$\phi_0, \ldots, \phi_{2I}, \quad \phi_{2I+1}(z) = \prod_{j=-I}^{I} (z - \zeta_j).$$

Problem

μ_k not computable from samples $x(k)$.

Hence Wiener-Levinson algorithm **not applicable**.

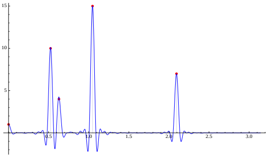
Idea

Replace μ by some discrete measure

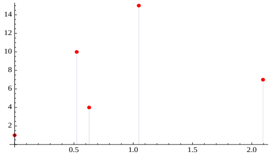
$$\nu^{(N)} = \sum_{j=-I}^{I} \lambda_{N,j} \, \delta_{\zeta_j}$$

s.t. related moments are computable.

OPUC II: Kernel-based Approach (Noiseless Recovery)



Kernel approximation to discrete measure



Modified discrete measure

For $N \geq 1$ consider suitable kernel

$$\Phi_N(t) = \sum_{m \in \mathbb{Z}} H\left(\frac{m}{N}\right) \mathrm{e}^{imt}$$

and let

$$w_{N,j} = \sum_{\ell=-I}^{I} a_j \overline{a_\ell} \frac{\Phi_N(\omega_j - \omega_\ell)}{\Phi_N(0)}.$$

Define

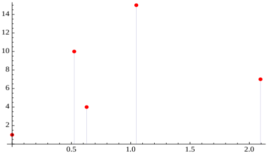
$$\lambda_{N,j} = \frac{w_{N,j} + \overline{w_{N,j}}}{2}$$

and let

$$\nu^{(N)} = \sum_{j=-I}^{I} \lambda_{N,j} \, \delta_{\zeta_j}.$$

OPUC II: Kernel-based Approach (Noiseless Recovery)

➤ Discrete measure



Modified discrete measure

Modified discrete measure

$$\nu^{(N)} = \sum_{j=-I}^{I} \lambda_{N,j}\, \delta_{\zeta_j}.$$

➤ Moments

For the moments sequence we have

(i)

$$\nu_k^{(N)} = \sum_{j=-I}^{I} \lambda_{N,j}\, \zeta_j^k = \frac{1}{\Phi_N(0)} \sum_{m\in\mathbb{Z}} H\Big(\frac{m}{N}\Big)\, x(m)\, \frac{x(m+k)+x(m-k)}{2}.$$

(ii) $\nu_k^{(N)}$ is real and $\nu_k^{(N)} = \nu_{-k}^{(N)}$,

and hopefully

$$\sum_n \sum_k c_n\, \overline{c_k}\, \nu_{n-k}^{(N)} \geq 0.$$

Theorem [Mhaskar, Prestin, F.]

Let $N \geq 1$ *be an integer. Let* $H : \mathbb{R} \to [0,1]$ *be an even* C^s *fct with* $H(t) = 0$ *for* $|t| \geq 1$ *. Further let*

$$\nu_k^{(N)} = \frac{1}{\Phi_N(0)} \sum_{m \in \mathbb{Z}} H\left(\frac{m}{N}\right) x(m) \, \frac{x(m+k) + x(m-k)}{2}.$$

Then we have

(i) *the moments* $\nu_k^{(N)}$ *are all real and* $\nu_k^{(N)} = \nu_{-k}^{(N)}, \; k \in \mathbb{Z}.$

(ii)

$$\nu_k^{(N)} = \sum_{k=-I}^{I} \lambda_{N,j} \, \mathrm{e}^{\mathrm{i}\omega_j k}, \quad \lambda_{N,j} = \mathfrak{Re}\left(a_k \sum_{\ell=-I}^{I} \overline{a_\ell} \, \frac{\Phi_N(\omega_j - \omega_\ell)}{\Phi_N(0)}\right).$$

(iii) *For* $N \geq \max\{2I+1, \left(\frac{\|H^{(2)}\|_\infty}{3\,\|H\|_1}\right)^{1/2}\}$ *we have*

$$\left|\lambda_{N,j} - |a_j|^2\right| \leq \frac{C\,|a_j]^2}{(N\,\mathfrak{q})^s}, \quad \text{and} \quad |\nu_k^{(N)} - \mu_k| \leq \frac{C}{(N\,\mathfrak{q})^s}.$$

In particular, if $N \geq C\,\mathfrak{q}^{-1}$ *then* $\{\nu_k^{(N)}\}_k$ *is positive definite.*

Here $\mathfrak{q} = \min_{j \neq k} |\omega_j - \omega_k|$ is the separation distance.

OPUC II: Kernel-based Approach (Noiseless Recovery)

➤ Related OPUC

Since $\{\nu_k^{(N)}\}_k$ is a positive definite sequence there exists a finite system of OPUC

$$\varphi_{N,0}, \ldots, \varphi_{N,2I}, \quad \varphi_{N,2I+1}(z) = \prod_{j=-I}^{I} (z - \zeta_j).$$

Moments $\{\nu_k^{(N)}\}_k$ are **computable**. Hence Verblunsky coefficients

$$\alpha_n = (-1)^n \frac{\det[(\nu_{j-\ell-1}^{(N)})_{j,\ell=0}^n]}{\det[(\overline{\nu_{j-\ell}^{(N)}})_{j,\ell=0}^n]}.$$

are computable.

Eigenvalues of related CMV matrix $C^{(2I+1)}$ give $\zeta_j,\ j = -I, \ldots, I$.

➤ **Example**

Define the kernel

$$\Phi_N(t) = \sum_{m\in\mathbb{Z}} H\left(\frac{m}{N}\right) e^{i\,m\,t}$$

by

$$H(t) := \begin{cases} 1, & \text{if } 0 \le t \le 1/2, \\ \exp\left(-\frac{\exp(2/(1-2t))}{1-t}\right), & \text{if } 1/2 < t < 1, \\ 0, & \text{if } t \ge 1. \end{cases}$$

Consider the **signal**

$$x(t) = \sin\left(\frac{\pi}{6}t\right) + \sin\left(\frac{\pi}{3}t\right) + \sin\left(\frac{\pi}{2}t\right) + 10\sin\left(\frac{3\pi}{4}t\right),$$

and samples $\{x(k)\}_{k=-199}^{199}$.

Results

ω_j	± 0.523489	± 1.047079	± 1.570719	± 2.356193
$\tilde{\omega}_j$	± 0.523598	± 1.047197	± 1.570796	± 2.356194

Outline

(1) The Frequency Detection Problem

(2) Classical Method: de Prony's Legacy

(3) OPUC I: Classical Approach

(4) OPUC II: Kernel-based Approach (Noiseless Recovery)

(5) **OPUC III: Kernel-based Approach (Recovery from Noisy Data)**

OPUC III: Kernel-based Approach (Recovery from Noisy Data)

➤ Statistical setting

We consider

$$\tilde{x}(k) = x(k) + \varepsilon(k), \quad x(t) = \sum_{j=-I}^{I} a_j \, \mathrm{e}^{i\omega_j t}$$

with the following assumptions regarding the noise

(a) $\{\varepsilon(k)\}_{k\in\mathbb{Z}}$ independent random variables with $\mathbb{E}[\varepsilon(k)] = 0$.

(b) $\varepsilon(k) \in [-\varepsilon, \varepsilon]$ almost everywhere for all $k \in \mathbb{Z}$.

(c) $\{\varepsilon(k)\varepsilon(\ell)\}_{k,\ell}$ independent random variables.

We **do not assume** that the random variables $\varepsilon(k)$ are **identically distributed**.

We write

$$\tilde{\varepsilon}_m(\ell) = \frac{1}{2}\{\varepsilon(m+\ell) + \varepsilon(m-\ell)\}$$

and

$$\sigma^2(k) = \mathrm{Var}\big[\varepsilon(k)\big] = \mathbb{E}\big[\{\varepsilon(k) - \mathbb{E}[\varepsilon(k)]\}^2\big] = \mathbb{E}[(\varepsilon(k))^2]$$

We assume $\sigma^2(k)$ is known or can be estimated by statistical methods.

➤ **Unbaised estimator**

Define

$$\tilde{\nu}_k^{(N)} := \frac{1}{\Phi_N(0)} \Big\{ \sum_{m\in\mathbb{Z}} H\big(\frac{m}{N}\big)\, \tilde{x}(k)\, \frac{\tilde{x}(m+k)+\tilde{x}(m-k)}{2} - \delta_{0,k} \sum_{m\in\mathbb{Z}} H\big(\frac{m}{N}\big)\, \sigma^2(m) \Big\}$$

Independence of $\varepsilon(k)$ implies

$$\mathbb{E}[\tilde{\nu}_k^{(N)}] = \mathbb{E}[\nu_k^{(N)}], \quad k = -N, \dots, N,$$

i.e. $\tilde{\nu}_k^{(N)}$ is an **unbaised estimator** for $\nu_k^{(N)}$.

OPUC III: Kernel-based Approach (Recovery from Noisy Data)

Theorem [Mhaskar, Prestin, F.]

Let $H : \mathbb{R} \to [0,1]$ *be an even* C^s *fct with* $H(t) = 0$ *for* $|t| \geq 1$ *. Let* $\alpha > 0$ *and* $N \in \mathbb{N}$ *with*

(1) $N \geq \max\left\{7, 2I+1, \left(\frac{\|H^{(2)}\|_\infty}{3\,\|H\|_1}\right)^{1/2}\right\}$

(2) $N \geq \frac{(2LM)^s}{\mathfrak{q}}$,

(3) $\frac{C(\varepsilon)}{(N\mathfrak{q})^s} \leq \left(\frac{(\alpha+2)\log N}{N}\right)^{1/2} \leq 1.$

Then with **probability exceeding** $1 - \frac{2}{N^\alpha}$ *, we have*

(i)

$$|\tilde{\nu}_k^{(N)} - \nu_k^{(N)}| \lesssim \left(\frac{(\alpha+2)\log N}{N}\right)^{1/2}, \quad k = -N, \dots, N,$$

(ii) $\{\tilde{\nu}_k^{(N)}\}_k$ *is a positive definite sequence.*

Hence the sequence of monic OPUC $\{\tilde{\varphi}_{N,n}\}_n$ *exists and, in particular,*

$$\tilde{\varphi}_{N,2I+1}(z) = \prod_{j=-I}^{I} (z - \tilde{\zeta}_j).$$

The points ζ_j, $j = -I, \dots, I$ can be estimated by the zero $\tilde{\zeta}_j$ of $\tilde{\varphi}_{N,2I+1}$.

➤ Numerical aspects

Since estimates of ζ_j are given by eigenvalues of CMV matrix $\tilde{\boldsymbol{C}}^{(2I+1)}$ we make use of

Theorem [Bauer-Fike]

Let $\boldsymbol{A}, \boldsymbol{E} \in \mathbb{C}^{n\times n}$ such that

$$\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

and let $\tilde{\lambda}_j$ be any eigenvalue of $\boldsymbol{A} + \boldsymbol{E}$. Then

$$\min_{1\le j\le n} |\lambda_j - \tilde{\lambda}_j| \le \|\boldsymbol{X}^{-1}\| \, \|\boldsymbol{X}\| \, \|\boldsymbol{E}\|.$$

➤ Numerical aspects

Theorem [Mhaskar, Prestin, F.]

Let $\alpha > 0$ and $N \in \mathbb{N}$ satisfying the conditions as above.

There is $c^\star > 0$ depending on μ, α, H, s satisfying the following condition with probability exceeding $1 - \frac{2}{N^\alpha}$

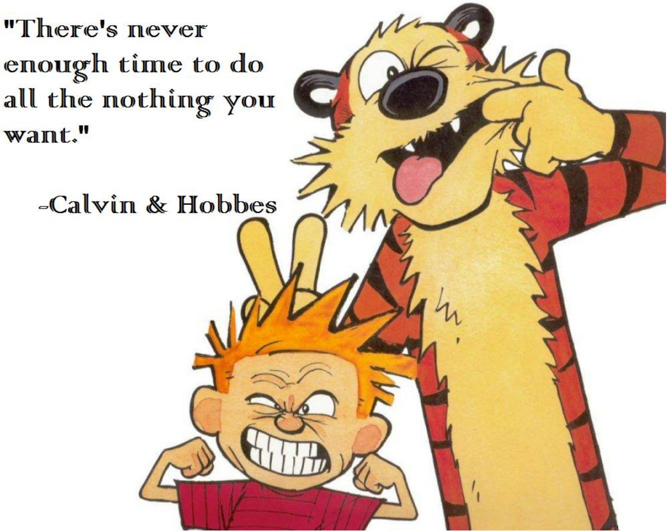
$$c^\star C(\varepsilon)\left(\frac{\log N}{N}\right)^{1/2} < \frac{\mathfrak{q}}{\pi},$$

where $C(\varepsilon)$ is a specific constant depending on ε.

For each ℓ there is exactly one point $\zeta_j,\ j = -I, \ldots, I$ in the disc

$$\left\{z \in \mathbb{C} : |z - \tilde{\zeta}_\ell| \leq c^\star C(\varepsilon)\big(\frac{\log N}{N}\big)^{1/2}\right\}$$

The End


"There's never enough time to do all the nothing you want."
-Calvin & Hobbes