A family of Non-Oscillatory, 6 points, Interpolatory subdivision scheme

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- $\chi^j \subset \chi^{j+1}$ nested grids. f^j is known data $\approx \chi^j$.
- $\mathcal{I}[x,\cdot]$ is a piecewise polynomial interpolatory technique:

  Generation of new data associated to χ^{j+1},

$$f_i^{j+1} = \mathcal{I}[x_i^{j+1}, f^j], \qquad \text{for } x_i^{j+1} \in \chi^{l+1}.$$

- From these reconstruction techniques, local rules derived.

- In general, the use of piecewise polynomial Lagrange interpolation based on l left points and r right points in a dyadic refinement framework leads to

Binary schemes:

$$\begin{cases} (S_{l,r}f)_{2i} & = f_i, \\ (S_{l,r}f)_{2i+1} & = \psi(f_{i-l}, \dots, f_{i+r-1}) = \sum_{k=-l}^{r-1} a_k^{l,r} f_{i+k}. \end{cases}$$

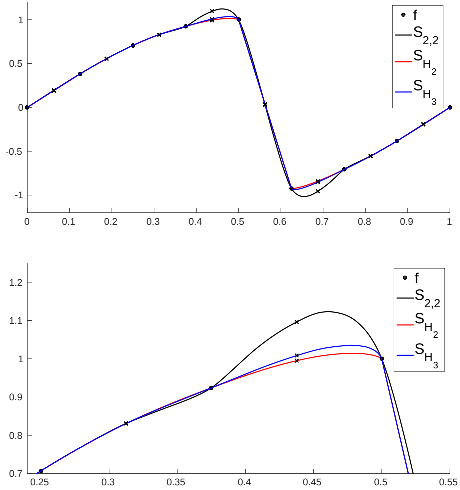
- Deslauries-Dubuc subdivision schemes: Interpolatory subdivision schemes related to piecewise polynomial interpolation based on a **centered stencil**, $l = r$.

- Lagrange Interpolation techniques do not preserve the shape properties of the original data to be refined, when the degree of the polinomial pieces is larger than 1.
- ENO-WENO, PPH subdivision. Nonlinear piecewise polynomial interpolatory techniques are used for avoid Gibbs-like behavior.

  [F. Arándiga. R.D. ... 1995-2000+], [A.Cohen, N.Dyn, B. Matei 2003], [S. Amat, R.D., J. Liandrat, J. Trillo 2006]
- Power_p schemes, Shape-preserving schemes: Nonlinear averages versus linear averages.

  [S. Amat, K. Dadourian, J. Liandrat 2011], [F. Kuijt, R. Van Damme 1999]

Nonoscillatory behavior of Power_p subdivision



4-point DD $\quad (S_{2,2}f)_{2n+1} = \frac{1}{2}(f_n + f_{n+1}) - \frac{1}{8}(\frac{1}{2}\nabla^2 f_{n-1} + \frac{1}{2}\nabla^2 f_n)$

Power$_p$ $\quad (S_{\mathrm{H}_p}f)_{2n+1} = \frac{1}{2}(f_n + f_{n+1}) - \frac{1}{8}\mathrm{H}_p(\nabla^2 f_{n-1}, \nabla^2 f_n)$

$$\mathrm{H}_p(x,y) = \frac{\mathrm{sgn}(x) + \mathrm{sgn}(y)}{2}\frac{x+y}{2}\left(1 - \left|\frac{x-y}{x+y}\right|^p\right)$$

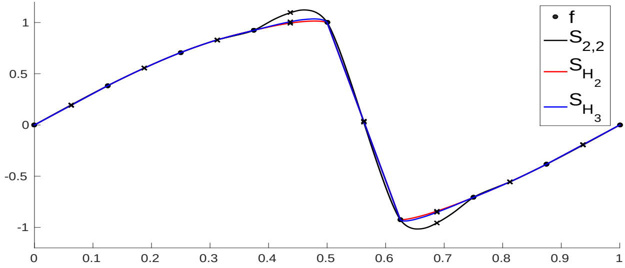
$$\mathrm{ave}_{\frac{1}{2},\frac{1}{2}}(x,y) = \frac{1}{2}x + \frac{1}{2}y$$

$$m \leq |\mathrm{ave}_{\frac{1}{2},\frac{1}{2}}(x,y)| \leq M,$$

$$m \leq |\mathrm{H}_p(x,y)| \leq \min\{M, pm\}$$

$$\begin{cases} m &= \min\{|x|,|y|\}, \\ M &= \max(|x|,|y|) \end{cases}$$

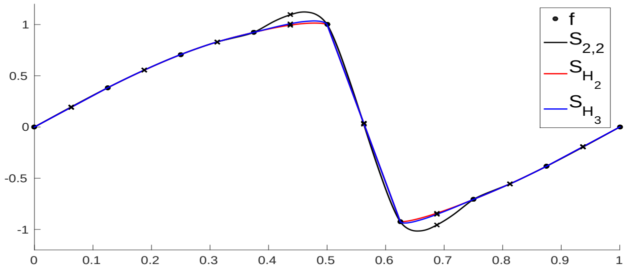
$$\begin{aligned} \mathrm{ave}_{\frac{1}{2},\frac{1}{2}}(\mathcal{O}(h^r),\mathcal{O}(h^s)) &= \mathcal{O}(h^{\min(r,s)}), \\ \mathrm{H}_p(\mathcal{O}(h^r),\mathcal{O}(h^s)) &= \mathcal{O}(h^{\max(r,s)}), \end{aligned} \quad r > 0, s > 0,$$



$f_i = F(x_i)$, $(x_{j+1} - x_j = h, \forall j)$
$F(x)$ piecewise-smooth, $\theta \in (x_m, x_{m+1})$ isolated discontinuity.

$$\nabla^2 f_j = \mathcal{O}(h^2), \quad j \neq m-1, m, \qquad \nabla^2 f_{m-1} = \mathcal{O}(1) = \nabla^2 f_m.$$

$$(S_{2,2}f)_{2j+1} = (S_{1,1}f)_{2j+1} + \mathcal{O}(h^2) \qquad j \neq m-1, \ldots, m+1$$
$$(S_{2,2}f)_{2j+1} = (S_{1,1}f)_{2j+1} + \mathcal{O}(1) \qquad j = m-1, m, m+1.$$



$f_i = F(x_i)$, $(x_{j+1} - x_j = h, \forall j)$
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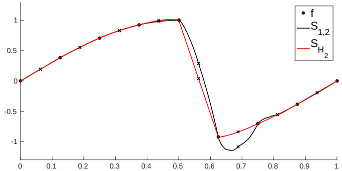
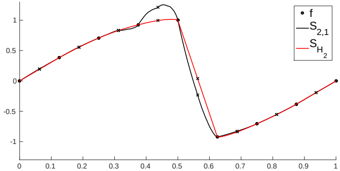
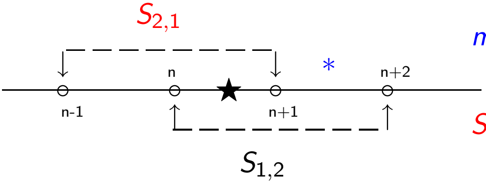
$$(S_{\mathrm{H}_p} f)_{2j+1} = (S_{1,1} f)_{2j+1} + \mathcal{O}(h^2) \qquad j \neq m.$$

Neville's algorithm for Lagrange interpolation:

$$S_{2,2} = \frac{1}{2} S_{2,1} + \frac{1}{2} S_{1,2} \quad \begin{cases} (S_{2,1}f)_{2n+1} = (S_{1,1}f)_{2n+1} - \frac{1}{8}\nabla^2 f_{n-1}, \\ (S_{1,2}f)_{2n+1} = (S_{1,1}f)_{2n+1} - \frac{1}{8}\nabla^2 f_{n}, \end{cases}$$





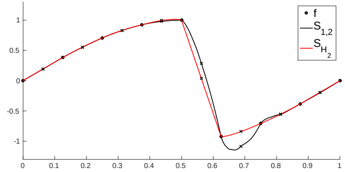


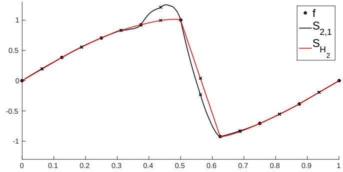
$$m \le |\mathrm{H}_p(x,y)| \le \min\{M, pm\}$$

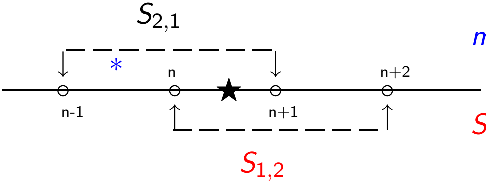
$$S_{\mathrm{H}_p} \approx \begin{cases} S_{2,1} & \text{around } \theta \\ & \end{cases}$$

Neville's algorithm for Lagrange interpolation:

$$S_{2,2} = \frac{1}{2}S_{2,1} + \frac{1}{2}S_{1,2} \quad \begin{cases} (S_{2,1}f)_{2n+1} = (S_{1,1}f)_{2n+1} - \frac{1}{8}\nabla^2 f_{n-1}, \\ (S_{1,2}f)_{2n+1} = (S_{1,1}f)_{2n+1} - \frac{1}{8}\nabla^2 f_{n}, \end{cases}$$







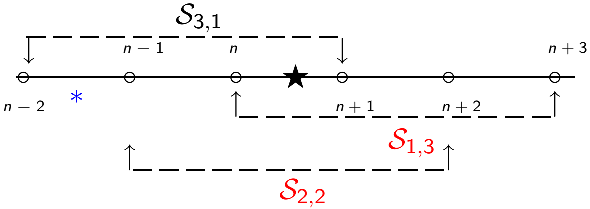
$$m \le |\mathrm{H}_p(x,y)| \le \min\{M, pm\}$$

$$S_{\mathrm{H}_p} \approx \begin{cases} \\ S_{1,2} \end{cases} \quad \text{around } \theta$$

General Neville: $S_{l,r} = \frac{r-1/2}{l+r-1} S_{l,r-1} + \frac{l-1/2}{l+r-1} S_{l-1,r}.$

For the 6-point DD linear scheme:

$$S_{3,3} = \frac{1}{2} S_{2,3} + \frac{1}{2} S_{3,2} = \frac{1}{2}(\frac{3}{8} S_{3,1} + \frac{5}{8} S_{2,2}) + \frac{1}{2}(\frac{3}{8} S_{1,3} + \frac{5}{8} S_{2,2}). \quad (1)$$



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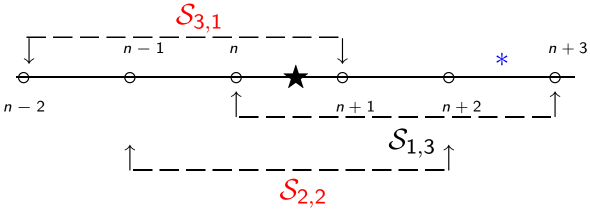

General Neville: $S_{l,r} = \frac{r-1/2}{l+r-1} S_{l,r-1} + \frac{l-1/2}{l+r-1} S_{l-1,r}.$

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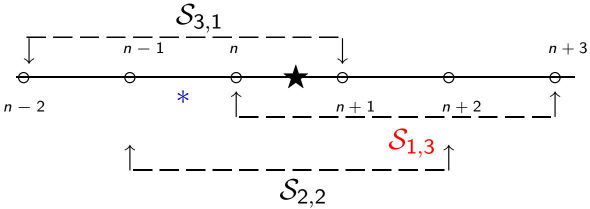
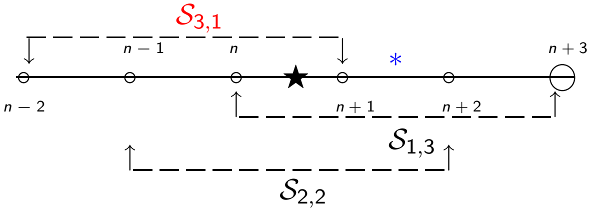


$$\text{General Neville:}\quad S_{l,r} = \frac{r-1/2}{l+r-1} S_{l,r-1} + \frac{l-1/2}{l+r-1} S_{l-1,r}.$$

For the 6-point DD linear scheme:

$$S_{3,3} = \frac{1}{2} S_{2,3} + \frac{1}{2} S_{3,2} = \frac{1}{2}\left(\frac{3}{8} S_{3,1} + \frac{5}{8} S_{2,2}\right) + \frac{1}{2}\left(\frac{3}{8} S_{1,3} + \frac{5}{8} S_{2,2}\right). \quad (4)$$



Required: Nonlinear analogs of $\text{ave}_{a,b}(x,y) = ax + by$

Weighted-Power$_p$ mean. $a > 0,\ b > 0\ a + b = 1,\ p \geq 1.$

$$\mathrm{W}_{p,a,b}(x,y) := \frac{sgn(x) + sgn(y)}{2}\,|ax + by|\left(1 - \frac{|x-y|^p}{(M + \frac{m}{\alpha})(M + \alpha m)^{p-1}}\right),$$

$M = \max\{|x|,|y|\},\ m = \min\{|x|,|y|\},\ \alpha = \max\{a,b\}/\min\{a,b\}.$

- Generalizes $\mathrm{H}_p(x,y)$: $\mathrm{W}_{p,\frac{1}{2},\frac{1}{2}}(x,y) = \mathrm{H}_p(x,y)$,
- Non-oscillatory: $\frac{1}{\alpha}m \leq |\mathrm{W}_{p,a,b}(x,y)| \leq p\alpha m$
- $\rightarrow\ \mathrm{W}_{p,a,b}(\mathcal{O}(h^r),\mathcal{O}(h^s)) = \mathcal{O}(h^{\max(r,s)})$

$$(S_{2,1}f)_{2n+1} = (S_{1,1}f)_{2n+1} - \frac{1}{8}\nabla^2 f_{n-1},$$
$$(S_{1,2}f)_{2n+1} = (S_{1,1}f)_{2n+1} - \frac{1}{8}\nabla^2 f_{n},$$

$$S_{l,r} = S_{1,1} + \mathcal{L}_{l,r} \circ \nabla^2, \quad \mathcal{L}_{l,r} \text{ linear operator } (\mathcal{L}_{l,r}f)_{2n} = 0.$$

$$S_{3,3} = \frac{1}{2}(\frac{3}{8}S_{3,1} + \frac{5}{8}S_{2,2}) + \frac{1}{2}(\frac{3}{8}S_{1,3} + \frac{5}{8}S_{2,2})$$

$$S_{3,3} = S_{1,1} + \text{ave}_{\frac{1}{2},\frac{1}{2}}[\text{ave}_{\frac{3}{8},\frac{5}{8}}(\mathcal{L}_{1,3},\mathcal{L}_{2,2}), \text{ave}_{\frac{3}{8},\frac{5}{8}}(\mathcal{L}_{3,1},\mathcal{L}_{2,2})] \circ \nabla^2,$$

Non-linear non-oscillatory version:

$$\mathrm{SHW}_{q,p} = S_{1,1} + \mathrm{H}_q[\mathrm{W}_{p,\frac{3}{8},\frac{5}{8}}(\mathcal{L}_{1,3},\mathcal{L}_{2,2}), \mathrm{W}_{p,\frac{3}{8},\frac{5}{8}}(\mathcal{L}_{3,1},\mathcal{L}_{2,2})] \circ \nabla^2).$$

$$(S_{2,1}f)_{2n+1} = (S_{1,1}f)_{2n+1} - \frac{1}{8}\nabla^2 f_{n-1},$$
$$(S_{1,2}f)_{2n+1} = (S_{1,1}f)_{2n+1} - \frac{1}{8}\nabla^2 f_{n},$$

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$$S_{3,3} = \frac{3}{8}(\frac{1}{2}S_{3,1} + \frac{1}{2}S_{1,3}) + +\frac{5}{8}S_{2,2}$$

$$S_{3,3} = S_{1,1} + \text{ave}_{\frac{3}{8},\frac{5}{8}}[\text{ave}_{\frac{1}{2},\frac{1}{2}}(\mathcal{L}_{1,3}, \mathcal{L}_{3,1}), \mathcal{L}_{2,2})] \circ \nabla^2,$$

Non-linear non-oscillatory version:

$$\mathrm{SWH}_{p,q} = S_{1,1} + \mathrm{W}_{p,\frac{3}{8},\frac{5}{8}}[\mathrm{H}_q(\mathcal{L}_{1,3}, \mathcal{L}_{3,1}), \mathcal{L}_{2,2})] \circ \nabla^2,$$

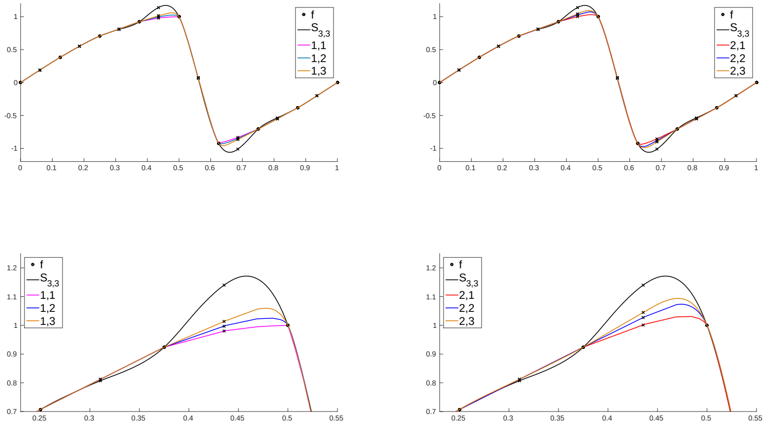
Thus, we consider the following two families of nonlinear schemes,

$$\mathrm{SWH}_{p,q} = S_{1,1} + \mathrm{W}_{p,\frac{3}{8},\frac{5}{8}}[\mathsf{H}_q(\mathcal{L}_{1,3},\mathcal{L}_{3,1}),\mathcal{L}_{2,2})] \circ \nabla^2,$$
$$\mathrm{SHW}_{q,p} = S_{1,1} + \mathsf{H}_q[\mathrm{W}_{p,\frac{3}{8},\frac{5}{8}}(\mathcal{L}_{1,3},\mathcal{L}_{2,2}),\mathrm{W}_{p,\frac{3}{8},\frac{5}{8}}(\mathcal{L}_{3,1},\mathcal{L}_{2,2})] \circ \nabla^2).$$

Nonlinear schemes of the form:

$$\forall f \in l^\infty(\mathbb{Z}), \qquad (S_{\mathcal{N}} f)_n = (S_{1,1} f)_n + \mathcal{F}(\nabla^2 f)_n, \quad \forall n \in \mathbb{Z} \qquad (5)$$

where $\mathcal{F} : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ is a nonlinear operator.


f
$S_{3,3}$
1,1
1,2
1,3
f
$S_{3,3}$
2,1
2,2
2,3
f
$S_{3,3}$
1,1
1,2
1,3
f
$S_{3,3}$
2,1
2,2
2,3

Proposition

The schemes $\mathrm{SWH}_{p,q}$, $\mathrm{SHW}_{q,p}$ *reproduce exactly* Π_3.

Linear subdivision schemes: exact polynomial reproduction guarantees existence of difference schemes.

$$S^{[l]} \circ \nabla^l = \nabla^l \circ S.$$

Nonlinear subdivision schemes: offset invariance guarantees existence of difference schemes.

Definition (Oswald-Harizanov, 2010)

A binary subdivision operator S *is offset invariant (OSI) for* Π_k *if for each* $f \in l_\infty(\mathbb{Z})$ *and any polynomial* $P(x) \in \Pi_m$, $m \leq k$ *there exists a polynomial,* Q, *of degree* $< m$ *such that*

$$S(f + P\big|_{\mathbb{Z}}) = Sf + (P + Q)\big|_{2^{-1}\mathbb{Z}}$$

Proposition

The schemes $\mathrm{SWH}_{p,q}$, $\mathrm{SHW}_{q,p}$ *are offset invariant for* Π_1.

There exist $S^{[1]}$ and $S^{[2]}$ for the new families of schemes.

$$S_{\mathcal{N}} = S_{1,1} + \mathcal{F} \circ \nabla^2, \tag{6}$$

Theorem (C1+ C2 → $S_{\mathcal{N}}$ is uniformly convergent)

$$\begin{array}{lll} \textbf{C1.} & \exists M > 0: & ||\mathcal{F}(f)||_\infty \leq M||f||_\infty, \\ \textbf{C2.} & \exists L > 0,\ 0 < T < 1: & ||\nabla^2 S_{\mathcal{N}}^L(f)||_\infty \leq T||\nabla^2 f||_\infty. \end{array}$$

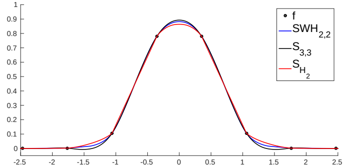
$$\nabla^2 S_{\mathcal{N}}^L = (S_{\mathcal{N}}^{[2]})^L \nabla^2$$

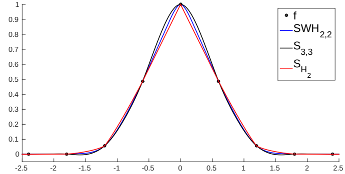
C2 $\equiv$ **C2':** $\exists L > 0,\ 0 < T < 1 : ||(S_{\mathcal{N}}^{[2]})^L(f)||_\infty \leq T||f||_\infty.$

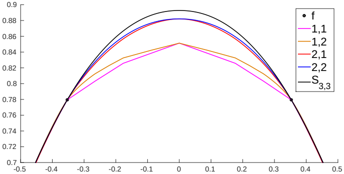
Theorem

$\mathrm{SWH}_{p,q}$ *and* $\mathrm{SHW}_{q,p}$ *are uniformly convergent, for all* $p, q \geq 1$.

$$||\mathrm{SWH}_{p,q}||_\infty, ||\mathrm{SHW}_{q,p}||_\infty < 1$$


f
SWH2,2
S3,3
SH2


f
SWH2,2
S3,3
SH2


f
1,1
1,2
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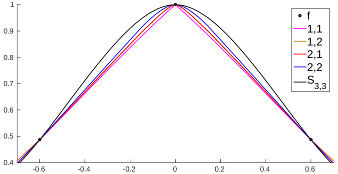
S3,3


f
1,1
1,2
2,1
2,2
S3,3

Numerical study of smoothness:

Assume $f(x) = S^\infty f^0 \in C^{r-}, l = [r]$, $0 \le \beta = r - l < 1$

$$\frac{f^{(l)}(x_{i+1}^k) - f^{(l)}(x_i^k)}{f^{(l)}(x_{i+1}^{k+1}) - f^{(l)}(x_i^{k+1})} \approx \frac{Ch_k^\beta}{Ch_{k+1}^\beta} = 2^\beta.$$

S interpolatory: $f_i^k = f(x_i^k)$, $f^{(l)}(x_i^k) \approx \nabla^l f_i^k/(h_k)^l = 2^{lk}\nabla^l f_i^k/h_0$,

$$\frac{\nabla^{l+1} f_i^k}{\nabla^{l+1} f_i^{k+1}} \approx 2^{l+\beta} \quad \rightarrow \quad l + \beta \approx \log_2\left(\frac{\|\nabla^{l+1} f^k\|_\infty}{\|\nabla^{l+1} f^{k+1}\|_\infty}\right),$$

$$\mathcal{R}_S^l = \log_2\left(\frac{\varrho_6^l}{\varrho_7^l}\right), \quad \varrho_k^l := \sup\{|(\nabla^{l+1} f^k)_i| : x_i^k \in [a, b]\}.$$

Left columns $[a, b] = [-0.1, 0.1]$. Right columns $[a, b] = [-3, 3]$.

l	$\mathcal{R}^l_{S_{3,3}}$		$\mathcal{R}^l_{\mathrm{SWH}_{1,2}}$		$\mathcal{R}^l_{\mathrm{SWH}_{2,1}}$		$\mathcal{R}^l_{\mathrm{SWH}_{2,2}}$		$\mathcal{R}^l_{S_{H_2}}$	
0.00	0.95	1.00	1.00	1.00	0.96	1.00	0.95	1.00	0.94	1.00
1.00	1.99	1.99	1.00	1.00	1.75	1.50	1.99	1.48	1.90	1.08
2.00	2.81	2.84	1.00	1.00	1.64	1.01	2.84	1.00	2.06	1.08
3.00	2.82	2.83	1.00	1.00	1.64	1.00	2.91	1.00	2.04	1.07
4.00	2.83	2.83	1.00	1.00	1.64	1.00	2.85	1.00	1.78	1.07

Table: Coarse Gaussian data (x=0 not in initial grid).

l	$\mathcal{R}^l_{S_{3,3}}$		$\mathcal{R}^l_{\mathrm{SWH}_{1,2}}$		$\mathcal{R}^l_{\mathrm{SWH}_{2,1}}$		$\mathcal{R}^l_{\mathrm{SWH}_{2,2}}$		$\mathcal{R}^l_{S_{H_2}}$	
0.00	0.91	1.00	1.00	1.00	1.00	1.00	0.95	1.00	1.00	1.00
1.00	1.99	1.99	1.69	1.69	1.44	1.44	1.93	1.93	1.00	1.00
2.00	2.82	2.82	1.63	1.63	1.48	1.48	2.47	1.34	1.00	1.00
3.00	2.83	2.83	1.63	1.63	1.48	1.48	2.58	1.27	1.00	1.00
4.00	2.83	2.83	1.38	1.38	1.47	1.47	2.64	1.30	1.00	1.00

Table: Coarse Gaussian data. (x=0 in initial grid).

$f_i = F(x_i)$, $x_{i+1} - x_i = h$, $\forall i$ F smooth,

Proposition ($r = 2p + 2$)

If $\nabla^2 f_n$ *have the same sign for each* n *and* $|F''(x)| > \rho > 0$

$$||S_{2,2}f - S_{H_p}f||_\infty = \mathcal{O}(h^r)$$

Proposition ($r = \min\{2p + 2, 3q + 2\}$)

If, for each n, $(\mathcal{L}_{l,r}f)_n$ *have the same sign, and* $|F''(x)| > \rho > 0$, *then*

$$||S_{3,3}f - \mathrm{SWH}_{p,q}f||_\infty = \mathcal{O}(h^r) = ||S_{3,3}f - \mathrm{SHW}_{q,p}f||_\infty$$

Definition (approximation order em after one iteration r)

$$||Sf - F|_{2^{-1}h\mathbb{Z}}||_\infty = O(h^r)$$

Corollary ($r = \min\{4, 2p + 2\}$)

If, for each n, $(\mathcal{L}_{l,r}f)_n$ *have the same sign, and* $|F''(x)| > \rho > 0$, *then*

$$||S_{H_p}f - F|_{2^{-1}h\mathbb{Z}}||_\infty = O(h^r)$$

Corollary ($r = \min\{6, 2p + 2, 3q + 2\}$)

If, for each n, $(\mathcal{L}_{l,r}f)_n$ *have the same sign, and* $|F''(x)| > \rho > 0$, *then*

$$||\mathrm{SWH}_{p,q}f - F|_{2^{-1}h\mathbb{Z}}||_\infty = O(h^r) = ||\mathrm{SHW}_{q,p}f - F|_{2^{-1}h\mathbb{Z}}||_\infty.$$

S a convergent subdivision scheme, $f_i = F(ih), i \in \mathbb{Z}$, $F(x)$ smooth

Definition (approximation order r)

$$||S^{\infty} f - F||_{\infty} = \mathcal{O}(h^r)$$

Proposition

S convergent subdivision scheme s.t.

$$||Sf - F|_{2^{-1}h\mathbb{Z}}||_{\infty} = \mathcal{O}(h^r)$$

then, if S stable

$$||S^{\infty} f - F||_{\infty} = \mathcal{O}(h^r)$$

$||S^\infty f^0 - F||_{L^\infty([a,b])} = \mathcal{O}(h^r)$: **Numerical study of r**

$E_S(h) := \max\{|(S^L f^0)_n - F(n2^{-L}h)|,\ n2^{-L}h \in [a,b]\} \approx ||S^\infty f^0 - F||_{L^\infty([a,b])}$

Gaussian data: $F(x) = e^{-2x^2}$, $h_0 = 0.1$ $L = 7$
Left column :$[a,b] = [-0.4, 0.4]$ $(|F''(x)| > \rho > 0)$
Right column: $[a,b] = [-1, -0.3]$, $(F''(0.5) = 0)$.

n	$E_{S_{3,3}}$	
0	3.4e-6	2.6e-6
1	5.7e-8	4.0e-8
2	9.0e-10	6.4e-10
3	1.4e-11	1.3e-11
r_n	5.95	5.86

r_n, the numerical order of accuracy= slope of the line obtained by linear regression of the data $(\log_2(h_n), \log_2(E_S(h_n)))$, $h_n = h_0/2^n$, Numerical Evidence: The order of approximation of the new schemes coincides with the order of approximation after one interation.

n	$E_{W(2,1)}$		$E_{W(2,2)}$		$E_{W(2,3)}$	
0	1.7e-5	1.5e-5	6.3e-6	8.9e-6	6.3e-6	8.7e-6
1	5.4e-7	5.3e-7	1.0e-7	2.1e-7	1.0e-7	2.1e-7
2	1.7e-8	1.7e-8	1.7e-9	5.7e-9	1.7e-9	5.6e-9
3	5.3e-10	5.3e-10	2.7e-11	1.5e-10	2.7e-11	1.5e-10
r_n	4.99	4.95	5.94	5.26	5.94	5.24
r_t	5	5	6	5	6	5

r_n= Numerical order of approximation ($S^\infty \approx S^7$)

r_t=Theoretical order of approximation after one iteration.
red r_t: from Corollary (only $|F''| > \rho > 0$).
blue r_t from direct Taylor expansions (if $(\mathcal{L}_{l,r}\nabla^2 f)_i$ do not change sign).
Similar conclusions for other data coming from smooth functions

Definition (Lipschitz Stability)

$$||S^j f - S^j g||_\infty \leq C||f - g||_\infty \qquad \forall f, g \in l_\infty(\mathbb{Z}) \quad \forall j \geq 0$$

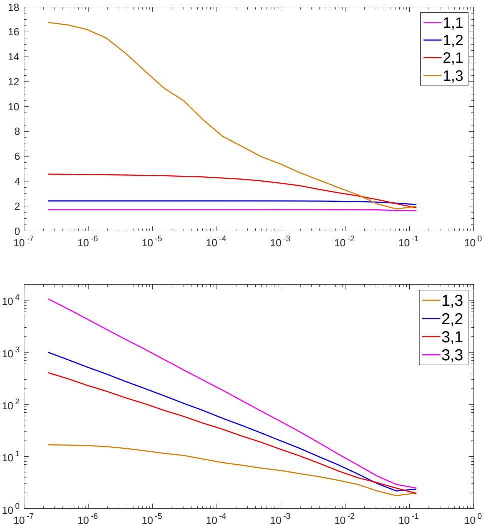
Numerical study of stability

$$C_S^j(h) = \sup_{||\theta||_\infty = 1} \frac{1}{h}||S^j(f^0 + h\theta) - S^j(f^0)||, \qquad h > 0, \tag{7}$$

(sup is taken over a large number of perturbations θ, $||\theta||_\infty = 1.$)

If S is stable, $C_S^j \leq C$, $\forall j$: any deviation with respect to this behavior is a sign of the unstability of the scheme.

Numerical tests: $f^0 = (-1, 0, 1, 1, -1, -1, -1, 1, 1)$, values of θ randomly chosen from the set $\{-1, 0, 1\}$.



Non-stable for $p + q > 3$!

$q \backslash p$	1	2	3
1	✓	✓	×
2	✓	×	×
3	×	×	×

$q \backslash p$	1	2	3
1	4	4	6
2	5	6	6
3	6	6	6

Table: Stability/accuracy persective on $\mathrm{SWH}_{p,q}$ and $\mathrm{SHW}_{q,p}$.

$$S_{\mathcal{N}} = S_{1,1} + \mathcal{F} \circ \nabla^2$$

Stability follows from the contractivity of some power of the second difference scheme.
Theoretical stability proofs in

- S. Harizanov, P. Oswald *Stability of nonlinear subdivision and multiscale transforms* Constr. Approx. 2010.
- F. Arandiga, R.D., M. Santagueda *The PCHIP Subdivision scheme* JCAM, 2016

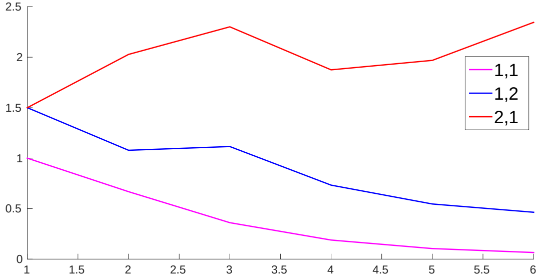
use the theory of Generalized Gradients of piecewise smooth Lipschitz functions/Generalized Jacobians of nonlinear subdivision schemes defined by such functions.

- $W_{p,a,b}(x, y)$ admits Generalized Gradients
- $\mathrm{SWH}^{[2]}_{p,q}$ admits a Generalized Jacobian,

but ....

Contractivity of $S^{[2]}$: Numerical study

$$T_S^j(h) \approx \sup_{\|\theta\|_\infty=1} \frac{1}{h}\|(S^{[2]})^j(f^0+h\theta)-(S^{[2]})^j(f^0)\|, \qquad h>0, \qquad (8)$$



- We have constructed two families of non-oscillatory subdivsion schemes that can be considered nonlinear/non-oscillatory versions of the 6-point Deslauries-Dubuc interpolatory subdivision scheme.
- Convergence ✓ (via second-difference scheme). Numerical study of regularity of limit function.
- Approximation properties. Order of approximation 5 ($p = 2, q = 1$). Order of approximation 6 for $p \geq 2, q \geq 2$ (Numerically)
- Stability. Some negative results. Some possibly possitive results hard to prove.

THANKS FOR YOUR ATTENTION!