Projection-based Parameter Estimation for Bivariate Exponential Sums

Benedikt Diederichs, Armin Iske

Department of Mathematics, University of Hamburg

Problem Formulation

We want to find the frequencies $\mathbf{y}_j \in \mathbb{R}^2$ and the corresponding amplitudes $c_j \in \mathbb{C} \setminus \{0\}$, $j = 1, ..., M$ of an exponential sum

$$f(\mathbf{x}) = \sum_{j=1}^{M} c_j e^{i\mathbf{y}_j \cdot \mathbf{x}} \qquad \text{for } \mathbf{x} \in \mathbb{R}^2.$$

Given: Order M of f and samples $f(\mathbf{k})$ taken on a finite set G.
Typical choice:

$$G_N := \{(k_1, k_2) \in \mathbb{Z}^2 \ : \ |k_1|, |k_2| \leq N\}.$$

We assume that $\mathbf{y}_j \in \mathbb{T}^2 = [0, 2\pi)^2$.

Univariate Problem

In the univariate case, i.e. $y_j \in [0, 2\pi)$,

$$f(x) = \sum_{j=1}^{M} c_j e^{iy_j x} \qquad \text{for } x \in \mathbb{R},$$

there are a number of efficient methods available (ESPRIT, OPUC, APM,...).

Given: Upper bound N of the order M and the samples

$$\{f(k),\ k = -N, ..., N\}$$

If y_j are well spaced, a stable reconstruction of the frequencies and the coefficients is possible.

Sampling along Lines

Idea: Sample f along a few lines $\ell_1, \ldots, \ell_L$, use a univariate method along these lines and combine the results to obtain an estimate for the frequency vectors of f.

Restricting f to a line

$$\ell_{\mathbf{v},b} = \{\lambda\mathbf{v} + b\boldsymbol{\eta} \mid \lambda \in \mathbb{R}\}$$

with $\mathbf{v} \perp \boldsymbol{\eta}$ unit vectors gives a univariate exponential sum

$$f|_{\ell_{\mathbf{v},b}}(\lambda\mathbf{v} + b\boldsymbol{\eta}) = \sum_{j=1}^{M} c_j e^{ib\mathbf{y}_j\cdot\boldsymbol{\eta}} e^{i\mathbf{y}_j\cdot\mathbf{v}\lambda} = \sum_{j=1}^{M_\ell} c_j^\ell e^{i\lambda y_j^\ell}.$$

Note that $M_\ell \leq M$.

Outline

Scattered Lines

We choose $\ell_1 = \ell_{\mathbf{v}_1, b_1}, \dots, \ell_L = \ell_{\mathbf{v}_L, b_L}$ with $\mathbf{v}_1, \dots, \mathbf{v}_L$ pairwise non-parallel.

Reconstruction Problem: Given $(y_j^{\ell_k}, c_j^{\ell_k})$, $j = 1, \dots, M_{\ell_k}$, $k = 1, \dots, L$ for which L can we calculate $\mathbf{y}_1, \dots, \mathbf{y}_M$?

Reformulation: Fix f. Consider

$$w : \mathbb{R}^2 \to \mathbb{C}, \quad w(\mathbf{x}) = \begin{cases} c_j & \text{if } \mathbf{x} = \mathbf{y}_j \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = \operatorname{supp} w$. We define the projection of w on $\ell_{\mathbf{v},b}$ by

$$w_{\mathbf{v},b}(x) = \sum_{\substack{\mathbf{y} \in X \\ \mathbf{v} \cdot \mathbf{y} = x}} w(\mathbf{y}) e^{ib\mathbf{y} \cdot \boldsymbol{\eta}}.$$

Projection of Point Clouds

It holds that

$$|\operatorname{supp} w_{\mathbf{v},b}| \leq |\mathbf{v} \cdot X| = |\{\mathbf{v} \cdot \mathbf{x} \ : \ \mathbf{x} \in X\}| \leq |X| = M.$$

Theorem (Renyi, 1952)

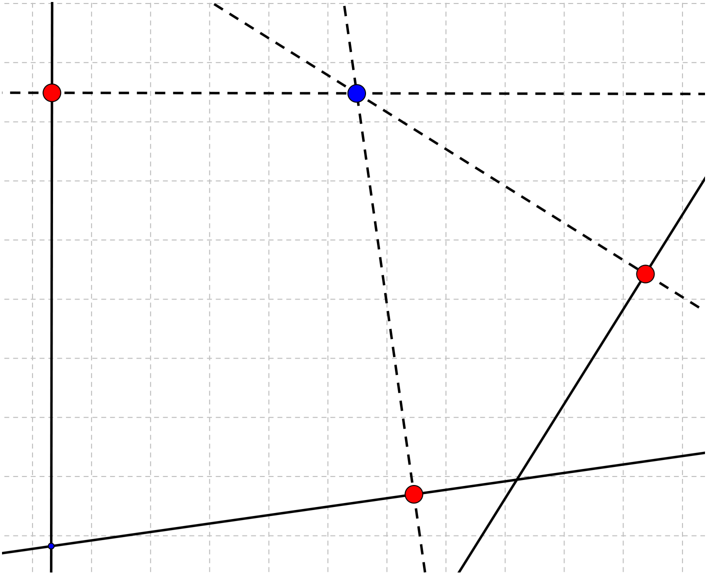
Assume $M+1$ projections $w_{\mathbf{v}_j,b_j}$, where $\mathbf{v}_j$ are pairwise linearly independent, are given and that $\operatorname{supp} w_{\mathbf{v}_j,b_j} = \mathbf{v}_j \cdot X$. Then w is uniquely determined.

Proof: Consider

$$\tilde{X} = \{\mathbf{x} \in \mathbb{R}^2 \ : \ \mathbf{v}_j \cdot \mathbf{x} \in \operatorname{supp} w_{\mathbf{v}_j,b_j} \text{ for all } j = 1, \ldots, M+1\}.$$

Then $X \subset \tilde{X}$.

Proof (continued): We show $\tilde{X} \subset X$.



Exponential Sums

Theorem (Potts, Tasche 2013)

Let G be a collection of points, suitable to apply *ESPRIT* along $M+1$ non-parallel lines $\ell_{\mathbf{v}_j,b_j}$. Let f be a bivariate exponential sum of order M. Denote the set of frequencies of f by X. Assume that

$$\mathbf{v}_j \cdot X = \{\text{Frequencies of } f_{\ell_{\mathbf{v}_j,b_j}}\}.$$

Then X can be calculated by

$$X = \{\mathbf{x} \in \mathbb{R}^2 \ : \ \mathbf{x}\cdot\mathbf{v}_j \text{ frequency of } f_{\ell_{\mathbf{v}_j,b_j}}\}.$$

This observation is the key point in the sparse approximate Prony method, presented in

Daniel Potts and Manfred Tasche. "Parameter estimation for multivariate exponential sums". In: *Electronic Transactions on Numerical Analysis* 40 (2013), pp. 204–224.

Projection of Point Clouds: Uniqueness

Is the condition supp $w_{\mathbf{v}_j,b_j} = \mathbf{v}_j \cdot X$ necessary?

Theorem (D., Iske, 2015)

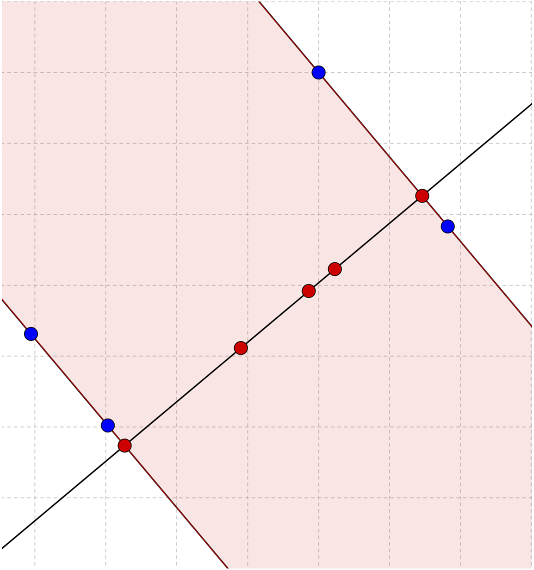
Let $w : \mathbb{R}^2 \to \mathbb{C}$, $w \neq 0$. If $w_{\mathbf{v}_j,b_j} = 0$ for $\mathbf{v}_1, \ldots, \mathbf{v}_L$ pairwise non-parallel, it holds that

$$|\text{supp}\, w| \geq 2L.$$

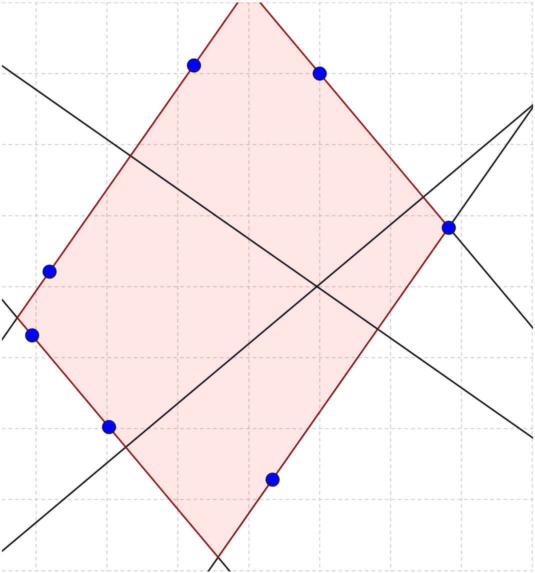
Corollary (D., Iske, 2015)

Any $w : \mathbb{R}^2 \to \mathbb{C}$ with $|\text{supp}\, w| \leq M$ is uniquely determined by its restriction on $M + 1$ non-parallel lines.

Proof (Theorem):



Proof (continued):



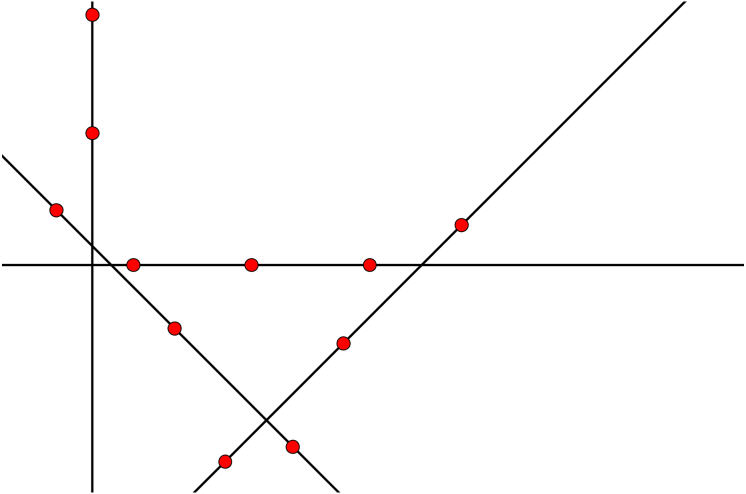
Lemma

Let $w : \mathbb{R}^2 \to \mathbb{C}$, $w \neq 0$ and let $w_{\mathbf{v}_j,b_j} = 0$ for $\mathbf{v}_1, \ldots, \mathbf{v}_L$ pairwise linearly independent be given. Then

1. $\tilde{X}_0$ contains $X = \operatorname{supp} w$, where
$$\tilde{X}_0 = \{\mathbf{x} \in \mathbb{R}^2 \ : \ \mathbf{x} \cdot \mathbf{v}_j \in \operatorname{supp} w_{\mathbf{v}_j,b_j} \text{ for two distinct } j\}.$$
2. Let $J = \{j \mid |\operatorname{supp} w_{\mathbf{v}_j,b_j}| \geq M - 1\}$. Then
$$\tilde{X}_1 = \{\mathbf{x} \in \tilde{X}_0 \ : \ \mathbf{x} \cdot \mathbf{v}_j \in \operatorname{supp} w_{\mathbf{v}_j,b_j} \text{ for all } j \in J\}$$
contains X.
3. Finally, X is contained in
$$\tilde{X}_2 = \{\mathbf{x} \in \tilde{X}_1 \ : \ \forall j \notin J \ \ (\mathbf{x} \cdot \mathbf{v}_j \in \operatorname{supp} w_{\mathbf{v}_j,b_j} \text{ or } \exists \mathbf{y} \in \tilde{X}_1, \ \mathbf{x} \neq \mathbf{y} \text{ with } \mathbf{x} \cdot \mathbf{v}_j = \mathbf{y} \cdot \mathbf{v}_j)\}.$$

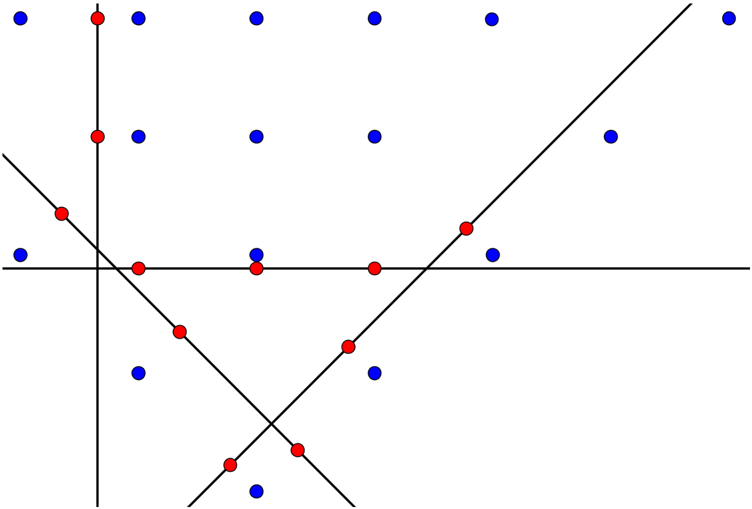
Example

Measurements



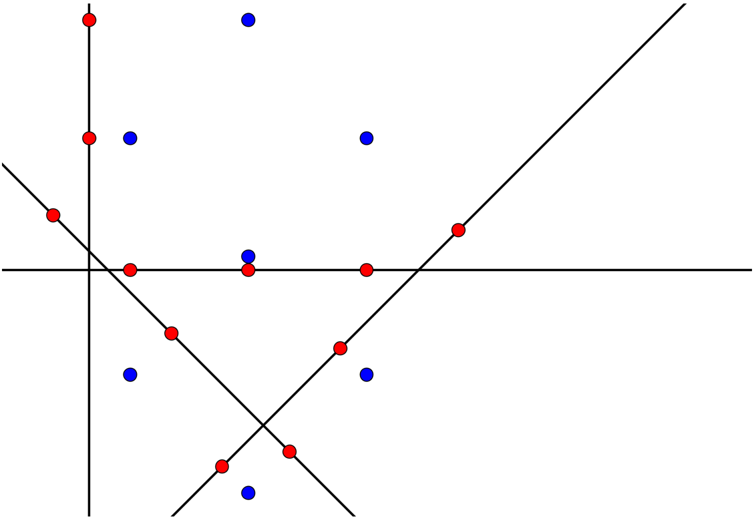
Example

$\tilde{X}_0$



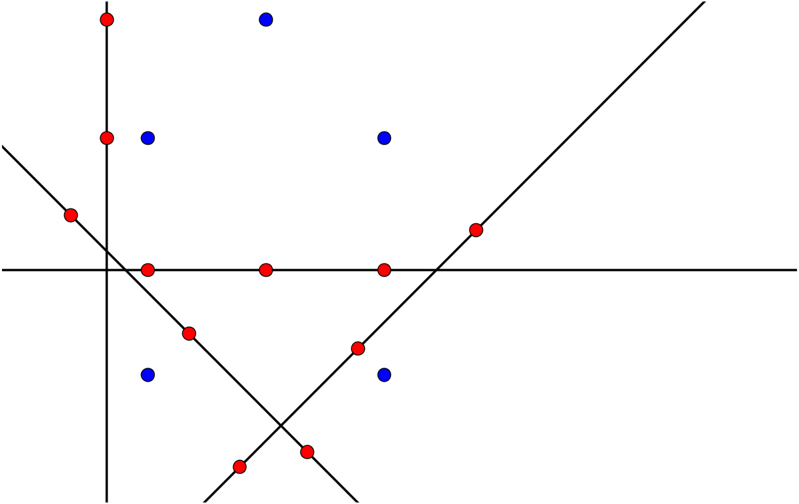
Example

$\tilde{X}_1$



Example

$\tilde{X}_2$



Back to Exponential Sums

Theorem (D., Iske, 2015)

Let G be a collection of points, suitable to apply *ESPRIT* along $M+1$ lines. Then the optimization problem

$$\min_{\mathbf{c}\in\mathbb{C}^{|\tilde{X}_2|}} \|\mathbf{c}\|_0$$

$$\text{subject to: } \sum_{\mathbf{y}\in\tilde{X}_2} c_y e^{i\mathbf{y}\cdot\mathbf{w}} = f(\mathbf{w}) \quad \forall \mathbf{w}\in G$$

has a unique solution $\mathbf{c} = (c_{\mathbf{y}})_{\mathbf{y}\in\tilde{X}_2}$. Moreover, $\{\mathbf{y}\in\tilde{X}_2 \mid c_y \neq 0\}$ are the frequency vectors of f and c_y are the corresponding coefficients.

Remarks:

- $|\tilde{X}_2| \leq cM^4$ is the best known bound.
- $\| \cdot \|_0$-minimization is NP-hard.
- The *restricted isometric property* is in general not satisfied.
- If the frequencies along one line are not correctly detected, the algorithm will break down.
- To obtain a stable reconstruction scheme, it seems necessary to assume well-separated frequency vectors and well-separated sampling points.

These results are published in

Benedikt Diederichs and Armin Iske. "Parameter estimation for bivariate exponential sums". In: *Proc. Sampling Theory and Applications* (2015), pp. 493 –497.

Parallel Lines

Parallel lines do not give multiple projections of the frequency vectors, but cancellation occurs less often.

$$f|_{\ell_1}(\lambda \mathbf{v} + b_1 \boldsymbol{\eta}) = \sum_{j=1}^{M} c_j e^{i b_1 \mathbf{y}_j \cdot \boldsymbol{\eta}} e^{i \mathbf{y}_j \cdot \mathbf{v} \lambda} = \sum_{j=1}^{M_1} c_j^{\ell_1} e^{i \lambda y_j^{\ell_1}}$$

$$f|_{\ell_2}(\lambda \mathbf{v} + b_2 \boldsymbol{\eta}) = \sum_{j=1}^{M} c_j e^{i b_2 \mathbf{y}_j \cdot \boldsymbol{\eta}} e^{i \mathbf{y}_j \cdot \mathbf{v} \lambda} = \sum_{j=1}^{M_2} c_j^{\ell_2} e^{i \lambda y_j^{\ell_2}}.$$

Note that for $b_1 \neq b_2$ it might happen that

$$\{y_j^{\ell_1} \ : \ j = 1, \ldots, M_1\} \neq \{y_j^{\ell_2} \ : \ j = 1, \ldots, M_2\}$$

Lemma

Let $\ell_{\mathbf{v},j} = \{\lambda \mathbf{v} + j\eta\}$, $j = 1, \dots, 2M$ be a family of parallel lines. Further, let f be an exponential sum of order M with frequencies X. Then

$$\bigcup_{j=1}^{2M} \{\text{Frequencies of } f|_{\ell_{\mathbf{v},j}}\} = \mathbf{v} \cdot X.$$

If f is sampled on $G_N = \{(m,n) \in \mathbb{Z}^2 \mid |m|, |n| \leq N\}$, we have samples along the lines

$$\ell_k^x = \{(x,k)^T,\ x \in \mathbb{R}\}, \quad \ell_k^y = \{(k,y)^T,\ y \in \mathbb{R}\}$$

where $k = -N, \dots, N$. Then we can construct

$$\tilde{X} = \{\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2 \ : \ x_1 \in \mathbf{e}_1 \cdot X,\ x_2 \in \mathbf{e}_2 \cdot X\} \supset X.$$

A Result on TV Minimization

Corollary (Candes, Fernandez-Granda, 2014)

An exponential sum f with frequency vectors satisfying the separation condition

$$\min_{\mathbf{y},\mathbf{y}'\in Y,\ \mathbf{y}\neq\mathbf{y}'} \|\mathbf{y}-\mathbf{y}'\|_{\mathbb{T}^2,\infty} \geq \frac{2.38}{N},$$

is the unique solution to

$$\min \|\mathbf{c}\|_1 \quad \text{subject to } \sum c_j e^{i\mathbf{g}\cdot\tilde{\mathbf{y}}_j} = f(\mathbf{g})\ \forall \mathbf{g} \in G_N.$$

Here, the minimization is carried out over all $M \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{R}^M$. The $\tilde{\mathbf{y}}_j$ in the constraint may be chosen arbitrarily for each $\mathbf{c}$.

Problem: This is an infinite dimensional optimization problem.

ℓ_1 Minimization

Using the result of Candes and Fernandez-Granda on TV-Minimization, we are able to reconstruct the frequency vectors by solving a minimization problem:

Theorem

Let f be a bivariate exponential sum of order M with well-separated frequencies. Then

$$\min_{\mathbf{c}\in\mathbb{C}^{|\tilde{X}|}} \|\mathbf{c}\|_1$$

$$\text{subject to: } \sum_{y\in\tilde{X}} c_y e^{i\mathbf{y}\cdot\mathbf{w}} = f(\mathbf{w}) \quad \forall \mathbf{w} \in G_N$$

has a unique solution $\mathbf{c} = (c_{\mathbf{y}})_{\mathbf{y}\in\tilde{X}}$. Moreover, $\{\mathbf{y} \in \tilde{X} \mid c_y \neq 0\}$ are the frequency vectors of f and c_y are the corresponding coefficients.

Simultaneous Frequency Estimation

If we implement this directly, we would have to

- use a univariate method along each of the $2N(2N+1)$ lines
- calculate the union of the x- and y- components of the frequencies
- calculate $\tilde{X}$
- solve the ℓ_1 minimization.

Better: Estimate the x (and y) components in a single step.

Example: Consider two exponential sums

$$f(x) = \sum_{j=1}^{M_1} c_j e^{iy_j x}, \qquad g(x) = \sum_{j=1}^{M_2} \tilde{c}_j e^{i\tilde{y}_j x}.$$

Let $M = |\{y_j\} \cup \{\tilde{y}_j\}| \leq M_1 + M_2$.

Given: $f(0), \dots, f(2N-1)$ and $g(0), \dots, g(2N-1)$ with $M \leq N$.

Simultaneous Estimation - ESPRIT

We can apply a modified ESPRIT algorithm. Let

$$\mathbf{H}_{P,Q} = (f(m+n-1))_{m=1,n=1}^{m=P,n=Q} \in \mathbb{C}^{P\times Q}$$
$$\tilde{\mathbf{H}}_{P,Q} = (g(m+n-1))_{m=1,n=1}^{m=P,n=Q} \in \mathbb{C}^{P\times Q}.$$

Let $N \le L \le M$. Let

$$\mathbf{G} = \begin{pmatrix} \mathbf{H}_{2N-L,L+1} \\ \tilde{\mathbf{H}}_{2N-L,L+1} \end{pmatrix}$$

We denote by $\mathbf{G}_0, \mathbf{G}_1$ the matrices we obtain by deleting the first resp. last column of $\mathbf{G}$. Then, the frequencies $\{y_j,\ \tilde{y}_j\}$ are the rank reducing numbers of the matrix pencil

$$\mathbf{G}_1 - \mu \mathbf{G}_0.$$

This can be reformulated as an eigenvalue problem.

A Closer Look on the Coefficients

We reconsider the restriction to the set of parallel lines $\ell_k^x = \{(x,k)^T,\ x \in \mathbb{R}\}$:

$$f|_{\ell_k^x}(x) = \sum_{j=1}^{M} c_j e^{ik(\mathbf{y}_j)_2} e^{ix(\mathbf{y}_j)_1} = \sum_{j=1}^{M_1} c_j(k) e^{ixy_{j,1}}.$$

We fix $j = 1, \ldots, M_1$. Let $\mathbf{y}_{n_1}, \ldots, \mathbf{y}_{n_{r_j}}$ be the frequency vectors with

$$(\mathbf{y}_{n_l})_1 = y_{j,1}, \quad l = 1, \ldots, r_j.$$

Then the coefficients are again an exponential sum:

$$c_j(k) = \sum_{l=1}^{r_j} c_{n_l} e^{ik(\mathbf{y}_{n_l})_2}$$

A New Algorithm

Let now $r = \max r_j$. Then each ℓ_k^x, $k \in \mathbb{Z}$ gives one value for each of the exponential sums $c_1, \dots, c_{M_1}$.

To use a univariate algorithm, we need $2r$ parallel lines. Then, we obtain frequencies and coefficients of the sums

$$c_j(x) = \sum_{l=1}^{r_j} c_{n_l} e^{ixy_l^{(j)}} = \sum_{l=1}^{r_j} c_{n_l} e^{ix(\mathbf{y}_{n_l})_2}.$$

The frequency vectors of f are then given by

$$\left\{ (y_{j,1}, y_l^{(j)})^T, \ j = 1, \dots, M_1, \ l = 1, \dots, r_j \right\}.$$

Projection-based Algorithm

Projection-based Algorithm

Given: $f(\mathbf{k}), \mathbf{k} \in G_N, \;\; N \geq M$

1. Apply a univariate method along $\ell_k^x, \; k = -N, \ldots, N$. Let

$$\{y_{j,1}, \; j = 1, \ldots, M_1\}$$

be the set of all frequencies, $c_j(k)$ be the corresponding coefficient along ℓ_k^x.

2. Apply a univariate method to $c_j(k), \;\; k = -N, \ldots, N$. Let $\{y_l^{(j)}, \;\; l = 1, \ldots, r_j\}$ denote the observed frequencies, $c_{j,l}$ the coefficients.

3. The frequency vectors and coefficients are given by

$$\left\{(c_{j,l}, (y_{j,1}, y_l^{(j)})^T), \; j = 1, \ldots, M_1, \;\; l = 1, \ldots, r_j\right\}.$$

Example

We consider $M = 3$ frequencies and let

$$f(\mathbf{x}) = \sum_{j=1}^{3} e^{i\mathbf{y}_j\cdot\mathbf{x}}.$$

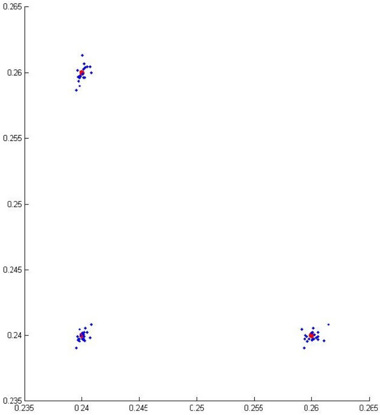
Then we take noisy samples, where we add equidistributed, independent noise in $[-1,1] + i[-1,1]$

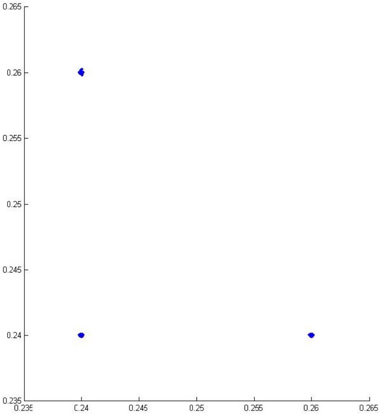
$$\tilde{f}(\mathbf{n}) = f(\mathbf{n}) + \epsilon(\mathbf{n})$$

This is a lot! Example:

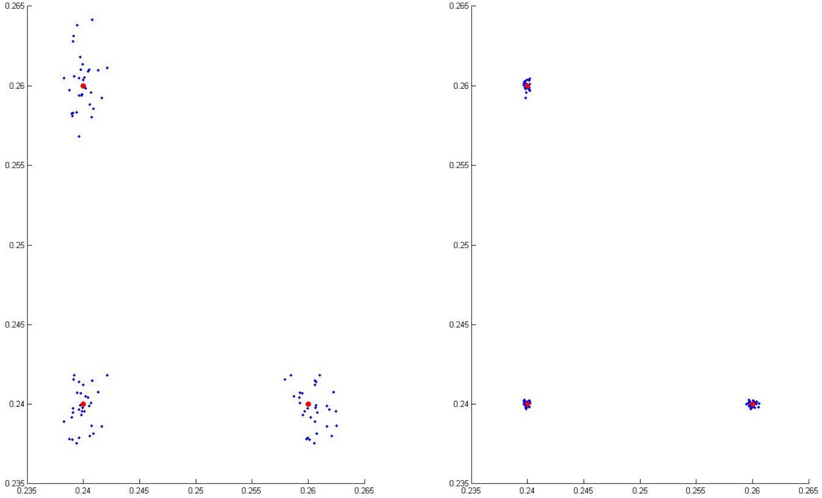
2.31-0.95i	0.97+2.24i	-2.17+0.98i
2.56-0.06i	1.09+2.98i	-2.29+0.29i

Result


0.265
0.26
0.255
0.25
0.245
0.24
0.235
0.235 0.24 0.245 0.25 0.255 0.26 0.265

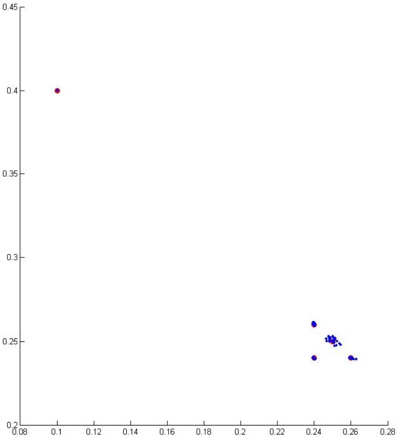

0.265
0.26
0.255
0.25
0.245
0.24
0.235
0.235 0.24 0.245 0.25 0.255 0.26 0.265

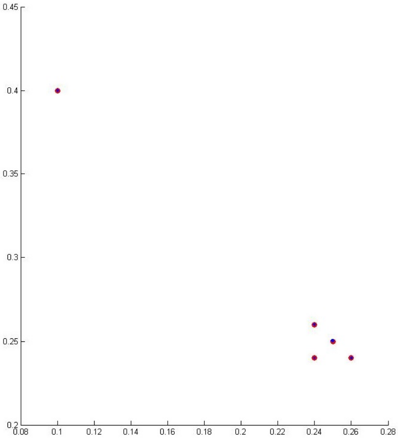
Result - Let's double the noise



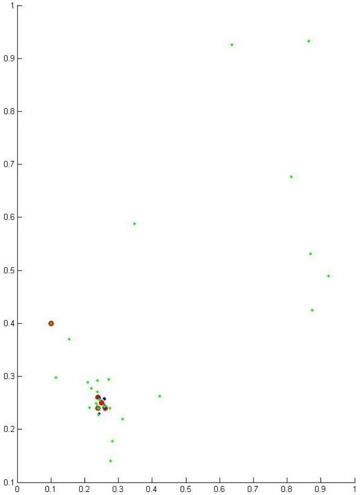
Example II

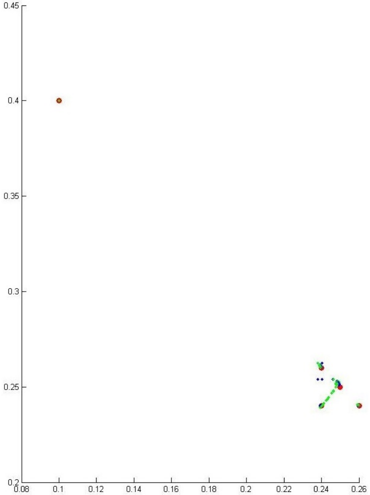
As a second example, let $M = 5$ and G_{40}, we add equidistributed noise in $0.3([-1,1] + i[-1,1])$.





Let's double the noise again





Example (Coefficient-based Method)

We test with randomly chosen frequencies and sample along

$$\{(m, j) \mid m = 0, \ldots, N-1, \; j = 0, 1\} \cup$$
$$\{(j, m) \mid m = 0, \ldots, N-1, \; j = 0, 1\}$$

Hence, we use $4N - 4$ samples.

Number of Freq.	N	Noise	Fails/100
5	15	0	0
5	15	10^{-4}	1
20	50	0	0
20	50	10^{-6}	18
50	150	0	3
50	150	10^{-8}	52

Thank you
for your attention!