Hermite subdivision schemes and polynomial-exponential reproduction

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2. Existence of *uniformly continuous* f_c such that

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3. *Nontriviality:* $f_c \neq 0$ for at least one c.

Extensions ...

Generalizations

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1. level-dependent schemes

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2. vector schemes

$$\boldsymbol{c}^n = S_{\boldsymbol{A}} \boldsymbol{c}^{n-1}, \quad n \in \mathbb{N},$$

with

$$\boldsymbol{c}^n = \left(\begin{bmatrix} c_0^n(\alpha) \\ c_1^n(\alpha) \\ \vdots \\ c_d^n(\alpha) \end{bmatrix} : \alpha \in \mathbb{Z} \right), \quad \boldsymbol{A} = \left(\begin{bmatrix} a_{00}(\alpha) & \dots & a_{0d}(\alpha) \\ \vdots & & \\ a_{d0}(\alpha) & \dots & a_{dd}(\alpha) \end{bmatrix} : \alpha \in \mathbb{Z} \right)$$

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Hermite subdivision:

Level-dependent + vector data (function & consecutive derivatives)

Rule

$$\boldsymbol{f}^n(\alpha) = (S^n_{\boldsymbol{A}} c)(\alpha)$$

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Level dependent subdivision

Preservation of polynomials

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In particular, necessary condition

$$S_a^n \text{ convergent} \quad \Rightarrow \quad S_a 1 = 1.$$

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Δ is *minimal*: $\quad S_g 1 = 0 \quad \Leftrightarrow \quad S_g = S_b \Delta.$

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Special sequences

$$C^d(\mathbb{R}) \ni f \mapsto \boldsymbol{v}_f := \left(\left[f^{(k)}(\alpha) \ : \ k = 0, \ldots, d\right], \alpha \in \mathbb{Z}\right)$$

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Spectral condition (Dubuc & Merrien)

There exist $p_j \in \Pi_j$, $j = 0, \dots, d$, such that

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Attention

d is the same!

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$$T_d := \begin{bmatrix} \Delta & -1 & \dots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & \Delta \end{bmatrix}$$

The Taylor Operator

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- $S_{\boldsymbol{C}} \boldsymbol{v}(\Pi_d) = \{0\} \quad \Leftrightarrow \quad S_{\boldsymbol{C}} = S_{\boldsymbol{B}} T_d$ (minimality)

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The Question

Yet another extension

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1. For $\Lambda \subset \mathbb{C}$ set

$$\begin{aligned} V_{d,\Lambda} &= \Pi_p \oplus \text{span}\,\{e^{\pm\lambda} \;:\; \lambda \in \Lambda\} \\ &= \text{span}\left\{1, x, \ldots, x^p, e^{\lambda_1 x}, e^{-\lambda_1 x}, \ldots, e^{\lambda_r x}, e^{-\lambda_r x}\right\} \end{aligned}$$

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4. Convergence? Factorization?

In search of ...

Annihilator - the pattern

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$$HS_{\boldsymbol{A}} = S_{\boldsymbol{B}} H$$

... implies factorization

Annihilator - Exponentials and polynomials

Characterization of minimal annihilators

An operator $H_{d,\Lambda}$ is a minimal cancellation operator for the space $V_{d,\Lambda} = \Pi_p \oplus \operatorname{span}\{e^{\pm\Lambda}\}$ iff its symbol satisfies

$$\boldsymbol{H}^*_{d,\Lambda}(z) = \begin{bmatrix} \boldsymbol{T}^*_p(z) & \boldsymbol{Q}^*(z) \\ 0 & \boldsymbol{R}^*(z) \end{bmatrix}$$

and

$$\boldsymbol{H}^*_{d,\Lambda}\left(e^{\pm\lambda}\right) \begin{bmatrix} 1 \\ \pm\lambda \\ \vdots \\ (\pm\lambda)^d \end{bmatrix} = 0, \qquad \lambda \in \Lambda.$$

(Symbol: $H^*(z) = \sum_\alpha H(\alpha) z^\alpha$)

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Example - Case $d = 3$

Space: $V_{3,\lambda} = \mathrm{span}\{1, x, e^{\lambda}, e^{-\lambda}\}$

$$H^*_{3,\lambda}(z) = \left[\begin{array}{cc|cc} z^{-1}-1 & -1 & \frac{2-e^{-\lambda}-e^{\lambda}}{2\lambda^2} & \frac{2\lambda+e^{-\lambda}-e^{\lambda}}{2\lambda^3} \\ 0 & z^{-1}-1 & \frac{e^{-\lambda}-e^{\lambda}}{2\lambda} & \frac{2-e^{-\lambda}-e^{\lambda}}{2\lambda^2} \\ \hline 0 & 0 & z^{-1}-\frac{e^{-\lambda}+e^{\lambda}}{2} & \frac{e^{-\lambda}-e^{\lambda}}{2\lambda} \\ 0 & 0 & \lambda\frac{e^{-\lambda}-e^{\lambda}}{2} & z^{-1}-\frac{e^{-\lambda}+e^{\lambda}}{2} \end{array}\right]$$

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Corollary

The Taylor operator is the minimal annihilator for "polynomials only".

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Corollary

The Taylor operator is the minimal annihilator for "polynomials only".

Remark

Proof needs *pairs* of frequencies $\pm\Lambda$.

Hermite subdivision **operators** preserving only polynomials

$$\tilde{S}_{\boldsymbol{A}^{[n]}} := \boldsymbol{D}^{-n-1} S_{\boldsymbol{A}} \boldsymbol{D}^{n}, \qquad n \in \mathbb{N}.$$

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3. In the limit: preservation of Π_d.

Definition: Weak contractivity

The scheme $\mathcal{S}(\boldsymbol{A}^{[n]} : n \geq 0)$ is said to be weakly contractive if

$$\sum_{n=0}^{\infty} \|\mathcal{S}_{\boldsymbol{A}^{[n]}} \, \mathcal{S}_{\boldsymbol{A}^{[n-1]}} \cdots \mathcal{S}_{\boldsymbol{A}^{[0]}}\| < \infty.$$

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Remark: generalization of classical contractivity in the case of level-independent schemes.

Definition: Convergence

C^d-convergence: existence of a uniformly continuous vector field $\phi : \mathbb{R} \to \mathbb{R}^{d+1}$, such that

1. $\lim_{n\to\infty} \sup_{\alpha\in\mathbb{Z}} \left\| \phi\left(2^{-n}\alpha\right) - \boldsymbol{f}^n(\alpha) \right\|_\infty = 0$

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Convergence result I

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the corresponding Hermite scheme $\mathcal{S}(\boldsymbol{A}^{[n]} : n \geq 0)$ *is* C^d*-convergent.*

Definition: Restricted convergence

Restricted C^k-convergence: existence of a uniformly continuous vector field $\boldsymbol{\phi} : \mathbb{R} \to \mathbb{R}^{d+1}$, such that

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the corresponding Hermite scheme $\mathcal{S}(\boldsymbol{A}^{[n]} : n \geq 0)$ *provides restricted* C^k*-convergence*

Work in progress/future work

1. Case of exponential polynomials
2. Several variables
3. (Multi)wavelet counterpart

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Thank you!