Canonical Quincunx Tight Framelets

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Outline

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Tight Framelets

1. M: $d \times d$ integer dilation matrix.
2. A tight M-framelet $\{\phi; \psi_1, \dots, \psi_L\}$ is such that

$$\|f\|^2_{L_2(\mathbb{R}^d)} = \sum_{k\in\mathbb{Z}^d} |\langle f, \phi(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle f, |\det(M)|^{j/2}\psi_\ell(M^j \cdot -k)\rangle|^2,$$

   for all $f \in L_2(\mathbb{R}^d)$.
3. M-refinable function ϕ:

$$\phi = |\det \mathrm{M}| \sum_{k\in\mathbb{Z}^d} a(k)\phi(\mathrm{M}\cdot -k)$$

4. Framelet functions ψ_ℓ:

$$\psi_\ell = |\det \mathrm{M}| \sum_{k\in\mathbb{Z}^d} b_\ell(k)\phi(\mathrm{M}\cdot -k)$$

Tight Framelet Filter Banks

1. Unitary Extension Principle: $a, b_1, \ldots, b_L \in l_0(\mathbb{Z}^d)$ with $\widehat{a}(0) = 1$. Then $\{\phi; \psi_1, \ldots, \psi_L\}$ is a tight M-framelet if and only if $\{a; b_1, \ldots, b_L\}$ is a tight M-framelet filter bank

$$\overline{\widehat{a}(\omega)}\widehat{a}(\omega + 2\pi\xi) + \sum_{\ell=1}^{L} \overline{\widehat{b_\ell}(\omega)}\widehat{b_\ell}(\omega + 2\pi\xi) = \delta(\xi),$$

where $\xi \in \Omega_{\mathsf{M}} := [(\mathsf{M}^{\mathsf{T}})^{-1}\mathbb{Z}^d] \cap [0, 1)^d$.

2. Low-pass filter: a. High-pass filters: $b_1, \ldots, b_L$.
3. $L = |\det \mathsf{M}| - 1$: Orthogonal M-wavelet low-pass filter,

$$\sum_{\xi \in \Omega_{\mathsf{M}}} |\widehat{a}(\omega + 2\pi\xi)|^2 = 1.$$

4. $L \geqslant |\det \mathsf{M}| - 1$: Non-orthogonal M-wavelet low-pass filter,

$$\sum_{\xi \in \Omega_{\mathsf{M}}} |\widehat{a}(\omega + 2\pi\xi)|^2 < 1.$$

Quincunx Tight Framelets

1. Quincunx dilation matrix $|\det M| = 2$. In 2D:

$$M_{\sqrt{2}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

2. Quincunx tight framelet filter bank:

$$|\widehat{a}(\omega)|^2 + \sum_{\ell=1}^{L} |\widehat{b_\ell}(\omega)|^2 = 1,$$

$$\overline{\widehat{a}(\omega)}\widehat{a}(\omega + (\pi, \pi)) + \sum_{\ell=1}^{L} \overline{\widehat{b_\ell}(\omega)}\widehat{b_\ell}(\omega + (\pi, \pi)) = 0.$$

Sum Rules, Linear Phase Moments, and Vanishing Moments

1. $\operatorname{sr}(u, \mathrm{M}) = n$: u has order n sum rules w.r.t. M if

$$\widehat{u}(\omega + 2\pi\xi) = \mathcal{O}(\|\omega\|^n), \omega \to 0, \ \forall \xi \in \Omega_{\mathrm{M}} \setminus \{0\}.$$

2. $\operatorname{lpm}(u) = n$: u has order n linear phase moments w.r.t. $c \in \mathbb{R}^d$ if

$$\widehat{u}(\omega) = e^{-ic\cdot\omega} + \mathcal{O}(\|\omega\|^n), \omega \to 0.$$

3. $\operatorname{vm}(u) = n$: u has order n vanishing moments if

$$\widehat{u}(\omega) = \mathcal{O}(\|\omega\|^n), \omega \to 0.$$

4. $\{a; b_1, \ldots, b_L\}$ is a tight M-framelet filter bank and a is symmetric about a point. Then

$$\min(\operatorname{vm}(b_1), \ldots, \operatorname{vm}(b_L)) = \min(\operatorname{sr}(a, \mathrm{M}), \frac{1}{2}\operatorname{lpm}(a)).$$

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Symmetry

1. G: group of $d \times d$ integer matrices, e.g.,

$$D_4 := \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

   Note that D_4 is compatible with $\mathrm{M}_{\sqrt{2}}$.

2. a is G-symmetric about a point $c \in \mathbb{R}^d$ if

$$a(E(k-c)+c) = a(k), \quad \forall k \in \mathbb{Z}^d;\ \forall E \in G.$$

Desirable Properties

A good tight framelet filter bank $\{a; b_1, \ldots, b_L\}$:

1. High order of vanishing moments for high-pass filters $b_1, \ldots, b_L$.
2. $\{a; b_1, \ldots, b_L\}$ possess symmetry property.
3. Number of filters L is small.
4. $b_1, \ldots, b_L$ have shortest possible support; should not be larger than that of a.
5. Regularity: Sobolev smoothness $\mathrm{sm}(a; \mathrm{M})$ could be arbitrarily large.

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Canonical Quincunx Tight Framelet Filter Banks

1. Quincunx tight framelet filter bank $\{a \equiv b_0; b_1, \ldots, b_{2s-1}\}$:

$$|\widehat{a}(\omega)|^2 + \sum_{\ell=1}^{L} |\widehat{b_\ell}(\omega)|^2 = 1, \quad \overline{\widehat{a}(\omega)}\widehat{a}(\omega + (\pi,\pi)) + \sum_{\ell=1}^{2s-1} \overline{\widehat{b_\ell}(\omega)}\widehat{b_\ell}(\omega + (\pi,\pi)) = 0.$$

2. $s = 1$: Orthogonal quincunx wavelet filter banks $\{a; b_1\}$ and

$$\widehat{b_1}(\omega_1, \omega_2) = e^{-i\omega_1} \overline{\widehat{a}(\omega_1 + \pi, \omega_2 + \pi)}.$$

3. $s = 2$: Double canonical quincunx wavelet filter banks $\{a; b_1, b_2, b_3\}$ with

$$\widehat{b_1}(\omega_1, \omega_2) = e^{-i\omega_1} \overline{\widehat{a}(\omega_1 + \pi, \omega_2 + \pi)},$$
$$\widehat{b_3}(\omega_1, \omega_2) = e^{-i\omega_1} \overline{\widehat{b_2}(\omega_1 + \pi, \omega_2 + \pi)}.$$

a is an non-orthogonal low-pass filter $\iff s \geqslant 2$.

Canonical Quincunx Tight Framelet Filter Banks (Cont')

1. *s*-multiple canonical quincunx tight framelet filter banks:

$$\widehat{b_{2\ell+1}}(\omega) = e^{-i\omega_1}\overline{\widehat{b_{2\ell}}(\omega + (\pi, \pi))}, \ \ell = 0, \ldots, s-1.$$

2. Double canonical and 6-multiple canonical quincunx tight framelet filter banks from tensor product of two 1D filter banks.

Theorem

$\{a; b_1, \ldots, b_{2s-1}\}$ is an s-multiple canonical quincunx tight framelet filter bank if and only if (SOS):

$$\sum_{\ell=1}^{s-1} |\widehat{b_{2\ell}}(\omega)|^2 + |\widehat{b_{2\ell}}(\omega + (\pi, \pi))|^2 = 1 - |\widehat{a}(\omega)|^2 - |\widehat{a}(\omega + (\pi, \pi))|^2$$

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A Family of Quincunx Low-Pass Filters

1. Deslauries-Dubuc interpolatory masks:

$$\widehat{a^{I}_{2n}}(\omega) := \cos^{2n}(\omega/2) \sum_{j=0}^{n-1} \binom{n-1+j}{j} \sin^{2j}(\omega/2).$$

2. 2D filters $\widehat{a^{2D}_{2n,2n}}(\omega_1,\omega_2) = \frac{1}{2}\left[\widehat{u}(\omega_1+\omega_2) + \widehat{u}(\omega_1-\omega_2)e^{-i\omega_2}\right]$, where $\widehat{u}(\omega) := (\widehat{a^{I}_{2n}}(\omega/2) - \widehat{a^{I}_{2n}}(\omega/2+\pi))e^{-i\omega/2}$.

Theorem (Han, Jiang, Shen, Z.)

$a^{2D}_{2n,2n}$ is the unique filter supported on $[1-n,n]^2 \cap \mathbb{Z}^2$, has order $2n$ sum rules w.r.t. $\mathsf{M}_{\sqrt{2}}$ and order $2n$ linear-phase moments with phase $c=(1/2,1/2)$. Moreover, $a^{2D}_{2n,2n}$ is real-valued and D_4-symmetric about $c=(1/2,1/2)$, which implies

$$\phi^{\mathsf{M}_{\sqrt{2}}}(E(\cdot-\mathbf{c}_\phi)+\mathbf{c}_\phi) = \phi^{\mathsf{M}_{\sqrt{2}}},\ \forall\, E \in D_4,$$

where $\mathbf{c}_\phi = (3/2,1/2)$ and $\widehat{\phi^{\mathsf{M}_{\sqrt{2}}}}(\omega) := \prod_{j=1}^{\infty} \widehat{a^{2D}_{2n,2n}}((\mathsf{M}^{\mathsf{T}}_{\sqrt{2}})^{-j}\omega)$.

L_2-Sobolev Smoothness

n	1	2	3	4	5	6	7	8	9
$\mathrm{sm}(a^{2D}_{2n,2n}, M_{\sqrt{2}})$	2.0	3.0365	3.5457	4.0269	4.4970	4.9658	5.4350	5.9038	6.3714
$\mathrm{sm}(a^{I}_{2n}, 2)$	1.5	2.4408	3.1751	3.7931	4.3441	4.8620	5.3628	5.8529	6.3352

Table: The smoothness exponents of $a^{2D}_{2n,2n}$ and a^{I}_{2n} for $n = 1, \dots, 9$.

Double Canonical Quincunx Tight Framelet Filter Banks

1. $a = a^{2D}_{2n,2n}$ with $n \in \mathbb{N}$. $\widehat{b_1}(\omega) := e^{-i\omega_1}\overline{\widehat{a}(\omega + (\pi,\pi))}$.
2. $\widehat{b_2}(\omega_1,\omega_2) := \frac{1}{2}[\widehat{v}(\omega_1+\omega_2) + \widehat{v}(\omega_1-\omega_2)e^{-i\omega_2}]$, where $\widehat{v}(\omega) := 2\widehat{a^D_n}(\omega/2)\widehat{a^D_n}(\omega/2+\pi)$ and a^D_n is the Daubechies othonormal filters.
3. $\widehat{b_3}(\omega) := e^{-i\omega_1}\overline{\widehat{b_2}(\omega + (\pi,\pi))}$

Theorem (Han, Jang, Shen, Z.)

$\{a = a^{2D}_{2n,2n}; b_1, b_2, b_3\}$ is a double canonical quincunx tight framelet filter bank satisfying

(1) *all high-pass filters b_1, b_2, b_3 have real coefficients and the following symmetry:*

$$b_1(E(k-\mathring{\mathbf{c}})+\mathring{\mathbf{c}}) = \det(E)b_1(k), \qquad \forall k \in \mathbb{Z}^2, E \in D_4 \text{ with } \mathring{\mathbf{c}} := (1/2,-1/2)$$
$$b_2(k_1, 1-k_2) = b_2(k_1,k_2) \text{ and } b_3(k_1,-1-k_2) = -b_3(k_1,k_2),\ \forall k_1,k_2 \in \mathbb{Z};$$

(2) *all high-pass filters b_1, b_2, b_3 have at least order n vanishing moments;*

(3) *the supports of b_1, b_2, b_3 are no larger than that of the low-pass filter a.*

Moreover, $\{\phi^{M_{\sqrt{2}}}; \psi_1, \psi_2, \psi_3\}$ is a tight $M_{\sqrt{2}}$-framelet in $L_2(\mathbb{R}^2)$ and

$$\psi_1(E(\cdot - \mathbf{c}_1)+\mathbf{c}_1) = \det(E)\psi_1, \qquad \forall E \in D_4 \quad \text{with} \quad \mathbf{c}_1 := (1,1),$$
$$\psi_2(x_2+1, x_1-1) = \psi_2(x_1,x_2), \qquad \psi_3(x_2,x_1) = -\psi_3(x_1,x_2).$$

General Construction

Theorem (Han, Jiang, Shen, Z.)

Let $u \in l_0(\mathbb{Z})$ be a finitely supported filter such that $|\widehat{u}(\omega)| \leqslant 1$. Define

$$\begin{aligned}
\widehat{a^{2D}}(\omega) &:= \frac{1}{2}\left[\widehat{u}(\omega_1+\omega_2)+\widehat{u}(\omega_1-\omega_2)e^{-i\omega_2}\right],\\
\widehat{b_1}(\omega) &:= e^{-i\omega_1}\overline{\widehat{a^{2D}}(\omega+(\pi,\pi))},\\
\widehat{b_2}(\omega) &:= \frac{1}{2}[\widehat{v}(\omega_1+\omega_2)+\widehat{v}(\omega_1-\omega_2)e^{-i\omega_2}],\\
\widehat{b_3}(\omega) &:= e^{-i\omega_1}\overline{\widehat{b_2}(\omega+(\pi,\pi))},
\end{aligned}$$

where $v \in l_0(\mathbb{Z})$ is a filter obtained from Fejér-Riesz Lemma and satisfying

$$|\widehat{v}(\omega)|^2 = 1 - |\widehat{u}(\omega)|^2.$$

Then $\{a^{2D}; b_1, b_2, b_3\}$ is a double canonical quincunx tight framelet filter bank. Moreover, there exist only Haar type double canonical quincunx tight framelet filter banks if u and v are required to both have symmetry.

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Multiple Canonical Quincunx Tight Framelet Filter Banks

1. $\{b_0; b_1, \ldots, b_L\}$ and $\{u_0; u_1, \ldots, u_K\}$ be 1D tight 2-framelet filter banks. Then

$$\{b_j \otimes u_k : 0 \leqslant j \leqslant L, 0 \leqslant k \leqslant K\}$$

is a quincunx tight framelet filter bank. However, NOT canonical.

2. Multiple canonical quincunx tight framelet filter banks:

Theorem (Han, Jiang, Shen, Z.)

Let $\{b_0; b_1, \ldots, b_{2s-1}\}$ is a one-dimensional s-multiple canonical tight 2-framelet filter bank:

$$\widehat{b_{2j+1}}(\omega) = e^{-i\omega}\overline{\widehat{b_{2j}}(\omega+\pi)}, \; j = 0, \ldots, s-1,$$

and $u_0, u_1, \ldots, u_L \in l_0(\mathbb{Z})$ are one-dimensional filters satisfying

$$|\widehat{u_0}(\omega)|^2 + |\widehat{u_1}(\omega)|^2 + \cdots + |\widehat{u_L}(\omega)|^2 = 1.$$

Define 2-D filters

$$\widehat{b^{2D}_{2j,k}}(\omega) := \widehat{b_{2j}}(\omega_1)\widehat{u_k}(\omega_2), \qquad \widehat{b^{2D}_{2j+1,k}}(\omega) := \widehat{b_{2j+1}}(\omega_1)\overline{\widehat{u_k}(\omega_2+\pi)},$$

for $j = 0, \ldots, s-1$ and $k = 0, \ldots, L$. Then $\{b^{2D}_{j,k} : j = 0, \ldots, 2s-1; k = 0, \ldots, L\}$ is an $s(L+1)$-multiple canonical quincunx tight framelet filter bank.

Symmetry

Theorem (Han, Jiang, Shen, Z.)

Let $\{b_0; b_1, \dots, b_{2s-1}\}$ is a one-dimensional s-multiple canonical tight 2-framelet filter bank:

$$\widehat{b_{2j+1}}(\omega) = e^{-i\omega}\overline{\widehat{b_{2j}}(\omega+\pi)},\ j = 0, \dots, s-1,$$

and $u_0, u_1, \dots, u_L \in l_0(\mathbb{Z})$ are one-dimensional filters satisfying

$$|\widehat{u_0}(\omega)|^2 + |\widehat{u_1}(\omega)|^2 + \cdots + |\widehat{u_L}(\omega)|^2 = 1.$$

Define 2-D filters

$$\widehat{b^{2D}_{2j,k}}(\omega) := \widehat{b_{2j}}(\omega_1)\widehat{u_k}(\omega_2), \qquad \widehat{b^{2D}_{2j+1,k}}(\omega) := \widehat{b_{2j+1}}(\omega_1)\overline{\widehat{u_k}(\omega_2+\pi)},$$

for $j = 0, \dots, s-1$ and $k = 0, \dots, L$. Then $\{b^{2D}_{j,k} : j = 0, \dots, 2s-1; k = 0, \dots, L\}$ is an $s(L+1)$-multiple canonical quincunx tight framelet filter bank.

1. Without symmetry: $s = 1$ and $L = 1 \Longrightarrow$ double canonical tensor product quincunx tight framelet filter banks.
2. With symmetry: $s > 1$ and $L > 1 \Longrightarrow$ smallest possible is 6-multiple canonical quincunx tight framelet filter banks.

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Tensor Product Double Canonical Family

Corollary (Han, Jiang, Shen, Z.)

Let a_n^D and a_m^D be the Daubechies orthogonal filters. Define

$$\widehat{a^{2D}}(\omega) := \widehat{a_n^D}(\omega_1)\widehat{a_m^D}(\omega_2), \qquad \widehat{b_1^{2D}}(\omega) := e^{-i\omega_1}\overline{\widehat{b_0^{2D}}(\omega + (\pi,\pi))},$$

$$\widehat{b_2^{2D}}(\omega) := \widehat{a_n^D}(\omega_1)\widehat{a_m^D}(\omega_2+\pi), \quad \widehat{b_3^{2D}}(\omega) := e^{-i\omega_1}\overline{\widehat{b_2^{2D}}(\omega + (\pi,\pi))}.$$

Then $\{a^{2D}; b_1^{2D}, b_2^{2D}, b_3^{2D}\}$ is a double-canonical quincunx tight framelet filter bank such that $\min(\mathrm{vm}(b_1^{2D}), \mathrm{vm}(b_2^{2D}), \mathrm{vm}(b_3^{2D})) \geqslant m+n$ and $\mathrm{sm}(a^{2D}, M_{\sqrt{2}}) \to \infty$ as $m+n \to \infty$.

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6-Multiple Canonical Construction with Symmetry

1. Given $a \in l_0(\mathbb{Z})$ satisfies
$$\widehat{w_1}(2\omega) = 1 - |\widehat{a}(\omega)|^2 - |\widehat{a}(\omega+\pi)|^2 \geqslant 0 \text{ and } \widehat{w_2}(\omega) = 1 - |\widehat{a}(\omega)|^2 \geqslant 0.$$
2. Fejér-Riesz Lemma:
$$|\widehat{v_1}(\omega)|^2 = \widehat{w_1}(\omega) \text{ and } |\widehat{v_2}(\omega)|^2 = \widehat{w_2}(\omega).$$
3. 1D canonical tight 2-framelet filter bank $\{b_0 = a; b_1, b_2, b_3\}$:
$$\widehat{b_1}(\omega) = e^{-i\omega}\overline{\widehat{a}(\omega+\pi)}$$
$$,\widehat{b_2}(\omega) := \frac{1}{2}(\widehat{v_1}(2\omega) + e^{-i\omega}\overline{\widehat{v_1}(2\omega)}),$$
$$\widehat{b_3}(\omega) := \frac{1}{2}(\widehat{v_1}(2\omega) - e^{-i\omega}\overline{\widehat{v_1}(2\omega)}).$$
4. 1D filters $\{u_0 = a; u_1, u_2\}$:
$$\widehat{u_1}(\omega) := \frac{1}{2}(\widehat{v_2}(\omega) + e^{-i\omega}\overline{\widehat{v_2}(\omega)}),$$
$$\widehat{u_2}(\omega) := \frac{1}{2}(\widehat{v_2}(\omega) - e^{-i\omega}\overline{\widehat{v_1}(\omega)}).$$
5. $\{b^{2D}_{j,k} : j = 0,1,2,3; k = 0,1,2\}$ defined as before is 6-multiple canonical quincunx tight framelet filter bank with symmetry.

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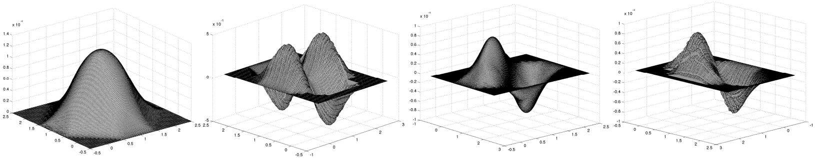
Double Canonical: Example 1

$\{a^{2D}_{2n,2n}; b_1, b_2, b_3\}$ with $n = 1$.

$$a = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ \boxed{1} & 1 \end{bmatrix}_{[0,1]^2}, \qquad b_1 = \frac{1}{4}\begin{bmatrix} \boxed{-1} & 1 \\ 1 & -1 \end{bmatrix}_{[0,1]\times[-1,0]},$$

$$b_2 = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ \boxed{1} & -1 \end{bmatrix}_{[0,1]^2}, \qquad b_3 = \frac{1}{4}\begin{bmatrix} \boxed{1} & 1 \\ -1 & -1 \end{bmatrix}_{[0,1]\times[-1,0]}.$$



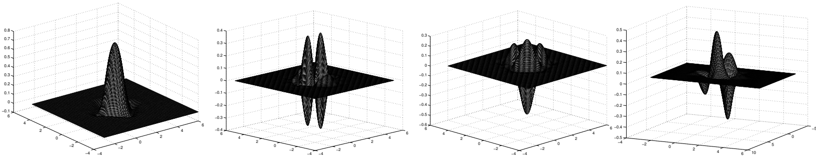
Double Canonical: Example 2

$\{a^{2D}_{2n,2n}; b_1, b_2, b_3\}$ with $n = 2$.

$$a = \frac{1}{32}\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 9 & 9 & 0 \\ 0 & \boxed{9} & 9 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}_{[-1,2]^2}, \quad b_1 = \frac{1}{32}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \boxed{-9} & 9 & 0 \\ 0 & 9 & -9 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}_{[-1,2]\times[-2,1]}$$

$$b_2 = \frac{1}{32}\begin{bmatrix} \sqrt{3}-2 & 0 & 0 & 2+\sqrt{3} \\ 0 & -\sqrt{3}+6 & -\sqrt{3}-6 & 0 \\ 0 & \boxed{-\sqrt{3}+6} & -\sqrt{3}-6 & 0 \\ \sqrt{3}-2 & 0 & 0 & 2+\sqrt{3} \end{bmatrix}_{[-1,2]^2},$$

$$b_3 = \frac{1}{32}\begin{bmatrix} -2-\sqrt{3} & 0 & 0 & \sqrt{3}-2 \\ 0 & \sqrt{3}+6 & \boxed{-\sqrt{3}+6} & 0 \\ 0 & -\sqrt{3}-6 & \sqrt{3}-6 & 0 \\ 2+\sqrt{3} & 0 & 0 & 2-\sqrt{3} \end{bmatrix}_{[-1,2]\times[-2,1]}.$$



Q & A

THANK YOU!

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