Interpolation with Multiquadrics

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Introduction

Radial Basis Functions (RBFs) approximate multivariate functions $f : \mathbf{R}^d \to \mathbf{R}$ by

$$s(x) = \sum_{\alpha \in A} \lambda_\alpha \phi(\|x - \alpha\|_2), \qquad x \in \mathbf{R}^d,$$

- the approximand is $f : \mathbf{R}^d \to \mathbf{R}$,
- $\phi : \mathbf{R}_+ \to \mathbf{R}$ is the **radial basis function (RBF)**,
- $A \subset \mathbf{R}^d$ is a (finite or infinite) discrete set of **centres**,
- $\| \cdot \|_2$ is the **Euclidean norm** (invariance, isotropy),
- the coefficients λ_α are often chosen such that the following **interpolation conditions** hold:

$$s(\alpha) = f(\alpha), \qquad \alpha \in A.$$

Several popular examples for RBF are

- $\phi(r) = \sqrt{r^2 + c^2}$, multiquadrics, Bernstein function, see e.g. (Beatson and Powell, 1992), (B., J. Jäger, A. Klein, W. Skrandies, 2016)
- $\phi(r) = \frac{1}{(r^2+c^2)^\beta}$, $c > 0$, $\beta = \frac{1}{2}$: inverse multiquadrics, $\beta = 1$: inverse quadratics,
- the Dagum class of completely monotonic (Mateu, Porcu, 09)

$$\phi(r) = 1 - \left(\frac{r^\beta}{1 + r^\beta}\right)^\gamma$$

  for $\beta \leq 1$ and $\beta\gamma \leq 1$ (sufficient conditions, Hofmann 2013).
- These are also available vector- and matrix-valued (Bevilacqua, B., Daley, Porcu, 2012).

Theorem (B., Dinew and Larsson, 2010)

If

$$\phi(r) = \sum_{k=0}^{\infty} a_k r^{2k}$$

converges locally uniformly and is such that $s_c(x)$ *using*

$$\phi_c(x) = \phi(x/c)$$

exists for all large c *and distinct nodes in* A *and* $a_k \neq 0$ *for infinitely many* k*, then*

$$\lim_{c \to \infty} s_c(x),$$

if it exists, does not depend on ϕ*.*

A further example.

- In order to obtain a radial basis function whose expansion contains only one zero term, we take $\phi(r) = \exp(-r) + r$, meaning that only a_1 equals zero.

- By regarding its Fourier transform in the distributional sense, we see that it is also conditionally negative definite:

- its Fourier transform is

$$\Gamma\left(\frac{d+1}{2}\right)2^d\,\pi^{(d-1)/2} \times \left(\frac{1}{(1+r^2)^{(d+1)/2}} - \frac{1}{r^{(d+1)}}\right)$$

  and, with $d = 1$, $2((1+r^2)^{-1} - r^{-2})$.

- The Fourier transform is < 0 for $r > 0$.

This is related to the RBF under tension (Bouhamidi, LeMéhauté, 2004, B., Derrien, LeMéhauté, 1995) which is a fundamental solution of the partial differential operator

$$\Delta^2 - \Delta,$$

namely

$$\begin{array}{rl} -0.5(\exp(-r)+r), & d=1, \\ -0.5(K_0(r)+\log(r))/\pi, & d=2, \\ -0.25(\exp(-r)-1)/(\pi r), & d=3. \end{array}$$

The semi-norm minimised by these splines is

$$(g,g)_{\mathcal{P}} = \sum_{|\gamma|=2} \frac{2}{\gamma!} \int_{\mathbf{R}^d} |D^\gamma g(x)|^2 \, dx + \sum_{|\gamma|=1} \int_{\mathbf{R}^d} |D^\gamma g(x)|^2 \, dx.$$

The semi-norm gives rise to a Pontryagin space, a variant of a Hilbert space with so-called negative squares. Specifically, the semi-inner product $(g, f)_{\mathcal{P}}$ is indefinite and there are nonzero "vectors" $\tilde{g}$ with $(\tilde{g}, \tilde{g})_{\mathcal{P}} \leq 0$.

This semi-inner product can also be expressed in the Gelfand-Naimark-Segal form

$$(g, f)_{\mathcal{P}} = \sum_{\alpha,\beta \in A} a_\alpha b_\beta \phi(\|\alpha - \beta\|_2).$$

We also generalise to kernels $K(x, y)$ which replace $\phi(\|x - y\|_2)$.

Convergence results for multiquadrics interpolation.

Usually, the multiquadrics interpolants are due to their conditionally positive definiteness used together with a constant (which however is not needed for well–poisedness of the interpolation problem, Micchelli (1986), Powell (1986)), so that the interpolant is in the Pontryagin native space $\mathcal{P}$:

$$s(x) = \sum_{\alpha \in A} \lambda_\alpha K(x, \alpha) + \hat{d}, \qquad x \in \mathbf{R}^d,$$

$$\sum_{\alpha \in A} \lambda_\alpha = 0.$$

The same is suitable for shifted thin-plate splines, except that linear polynomials are added and the moment conditions are of order one.

Convergence Theorem.

Several convergence order results are known: Schaback and Wu (1992), B. and Dyn [for gridded data] (1993):

Theorem

Let the constant $c > 0$ be independent of the spacing of the centres in A. Let the RBF be the multiquadrics. Let the distance from one centre to one its nearest neighbour be at most a fixed multiple of $h > 0$. Then, the uniform error of the radial basis function interpolant is at most $O(h^\ell)$ as long as the approximand is at least of smoothness of order ℓ and in the native space $\mathcal{P}$.

(B. and Davydov, 2016) generalised this theorem to multiquadrics without the constant added:

Theorem (B., Davydov, 2016)

Assume that K is a symmetric, conditionally positive definite kernel of order one on $\mathbf{R}^d$. Denote by $\mathcal{P}$ the native Pontryagin space of K on a set $\Omega \subset \mathbf{R}^d$ and assume in addition that $K(x,x) \leq -1$ for all $x \in \Omega$. Let $A = \{\alpha_1, \ldots, \alpha_N\} \subset \Omega$ and $x \in \Omega$. Then for any $f \in \mathcal{P}$ and any q such that $\partial^{\gamma,\beta} K \in C(\mathbf{R}^d \times \mathbf{R}^d)$ for all $|\gamma|, |\beta| \leq q$,

$$|f(x) - s(x)| \leq \rho(x,A)\, \tilde{M}_{K,q}^{1/2} \Big((f,f)_{\mathcal{P}} + \frac{1}{N} \sum_{j=1}^{N} |f(\alpha_j)|^2 \Big)^{1/2},$$

where

$$\rho(x,A) = \sup \Big\{ p(z) \mid p \in \mathbf{P}_d^q, |p(\alpha_i)| \leq \|z - \alpha_i\|^q \ \forall\, i \Big\}.$$