Numerical inpainting techniques
–
Modeling with inherent discretization

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joint work with Tomas Sauer

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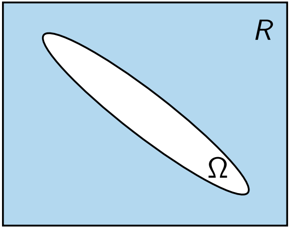
Inpainting: name origin

source: http://www.inrestauro.wordpress.com

Inpainting model

Given:

- domain of image R
- hole $\Omega \subset R$
- data/function g over $R \setminus \Omega$



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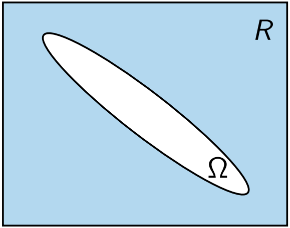
- $u = g$ outside of Ω
- $u|_{\Omega}$ "suitable" extension



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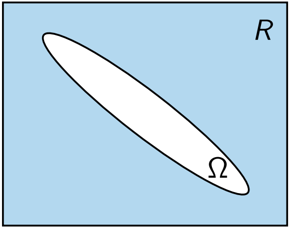
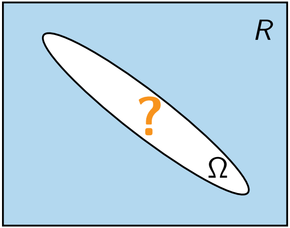
$\Rightarrow$ **interpolation problem**

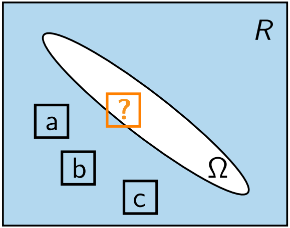


Inpainting methods



Inpainting methods

- pixel-/patch-based



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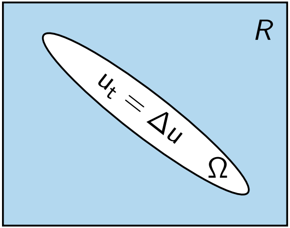
- pixel-/patch-based
- PDE-based/variational approaches



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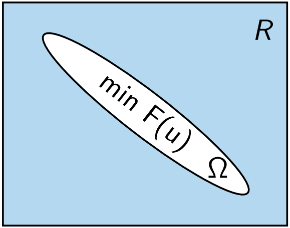
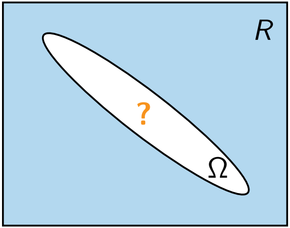


Inpainting methods

- pixel-/patch-based
- PDE-based/variational approaches
- combinations of methods
- ...



Inpainting methods

- pixel-/patch-based $\Rightarrow$ textured images
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Inpainting methods

- pixel-/patch-based $\Rightarrow$ textured images
- PDE-based/variational approaches $\Rightarrow$ structured/geometric images
- combinations of methods
- ...

Overview

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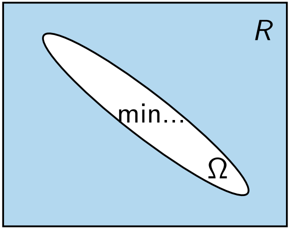
1. Variational approaches
2. Modeling and discretization
3. Implementation
4. Example

Variational approaches

Minimization of

1. constrained problem: $F(u)$ such that $u = g$ on B
2. unconstrained problem: $F(u) + \lambda \|u - g\|_B$



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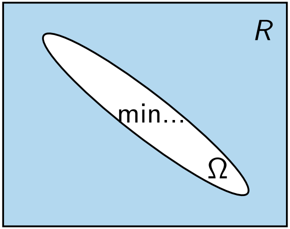

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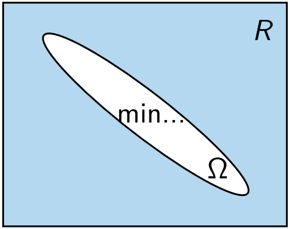

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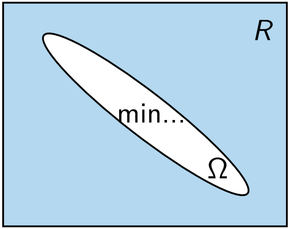

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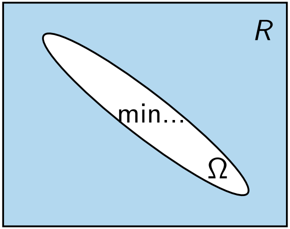

We can choose:

- $B \subseteq R \setminus \Omega$
- $\|\cdot\|_B$
- $F(u) = \int\limits_{U(\Omega)} \Phi(u)\, d\mathbf{x}$

TV minimization (Chan and Shen[1])

$$F(u) = \int_{U(\Omega)} |\nabla u| \, d\mathbf{x}, \quad u = g \text{ on } B = U(\Omega) \setminus \Omega,$$

with $|\nabla u|^2 = (\partial_1 u)^2 + (\partial_2 u)^2$.

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$$\frac{1}{|B|} \int_B |u - g|^2 \, d\mathbf{x} = \sigma^2$$

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- $\Omega = \emptyset$: TV restoration model Rudin, Osher and Fatemi[2]

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[2] Rudin et al.: "Nonlinear total variation based noise removal algorithms", *Physica D*, 1992.

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- $\partial_1 u_{i,j} := u_{i+1,j} - u_{i,j}$
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Discrete constraint TV inpainting

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} \|\nabla_{i,j} u\|_2 \\ \text{such that} \quad & u_{i,j} = g_{i,j} \quad \text{on } B \end{aligned}$$

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Standard approaches: $u \in BV(R)$

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Idea: Model problem with tensor product splines

Modeling with tensor product splines

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Modeling with tensor product splines

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   - adaptable to data
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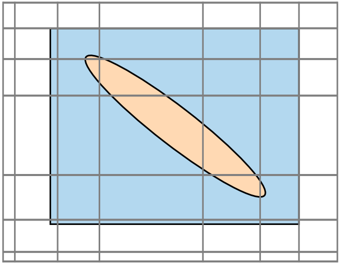
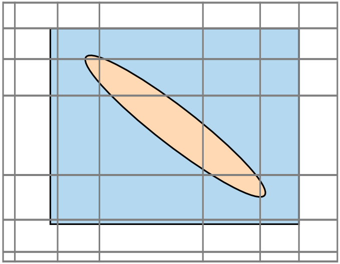
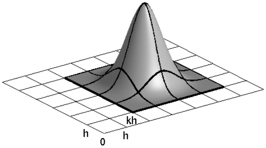
2. How to choose the grid?
3. How to model the problem?

Step 1: How to choose the grid?



- non-uniform knots $T := T_1 \otimes T_2$
- grid width $\mathbf{h} = (h_1, h_2)$





- non-uniform knots $T := T_1 \otimes T_2$
- grid width $\mathbf{h} = (h_1, h_2)$
- order $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$
- tensor product B-splines $b_{\mathbf{k}}(\mathbf{x})$, $\mathbf{k} \in K$ and $\mathbf{x} \in R$
- tensor product spline space $\mathcal{S}_{\mathbf{n}}(T, R)$

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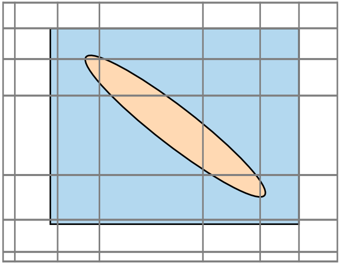

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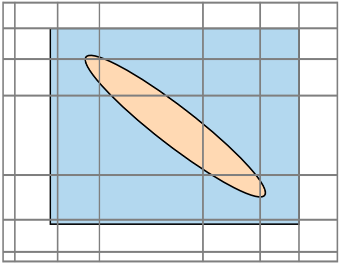

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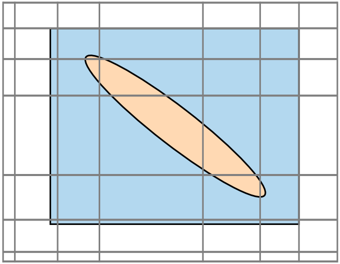

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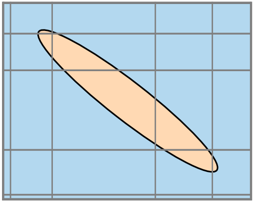

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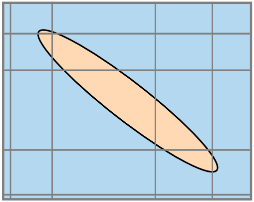

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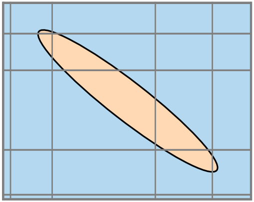

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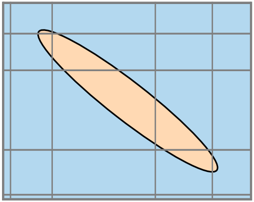

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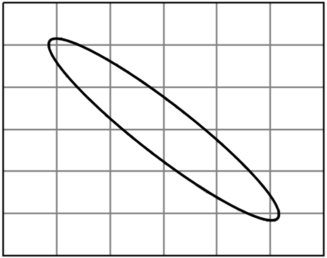
$\Rightarrow$ Determine $\mathbf{f} := (f_{\mathbf{k}})_{\mathbf{k} \in K}$!

Step 2: side conditions

Reproduce image g over $R^* := R \setminus \Omega$



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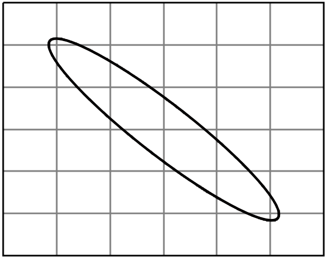

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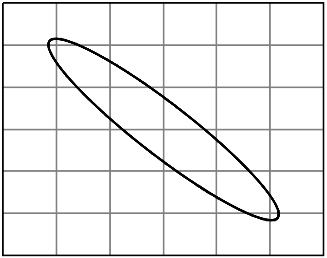

Spline interpolation at $\mathbf{x_k}$, $\mathbf{k} \in K$:

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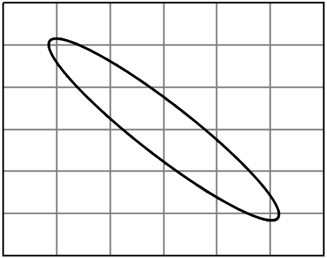

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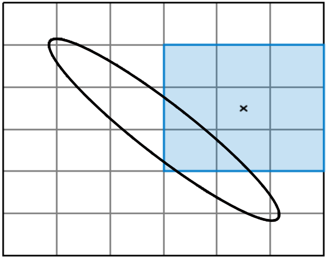

order $\mathbf{n} = (3, 3)$

Spline interpolation at $\mathbf{x_k}$, $\mathbf{k} \in K$:

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- fulfilled for Greville abscissae $\xi_{\mathbf{k}} \in \Xi$, $\mathbf{k} \in K$

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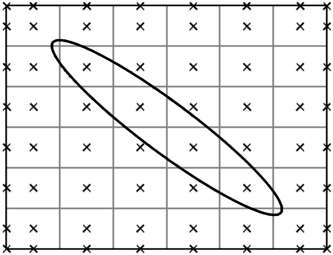
Greville abscissae



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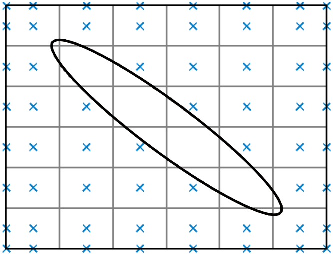

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Reproduce image g over $R^* := R \setminus \Omega$ by interpolation:

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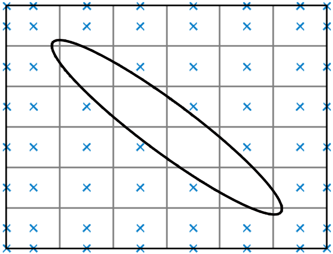

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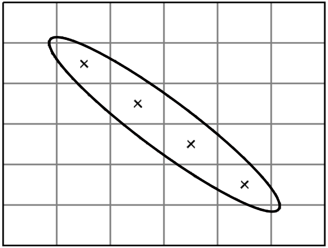

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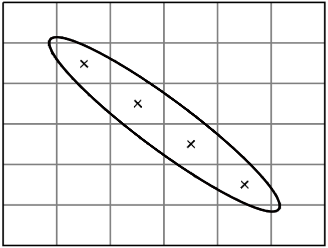

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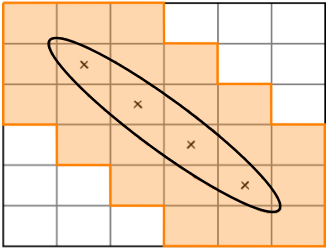

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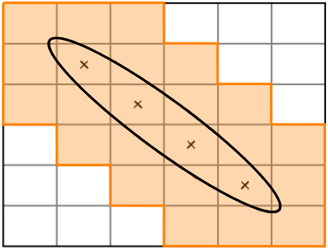

order $\mathbf{n} = (3, 3)$

- Greville abscissae $\xi_{\mathbf{k}} \in \Omega$
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Step 3: TV minimization

Minimization over u:

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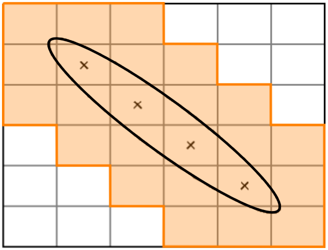

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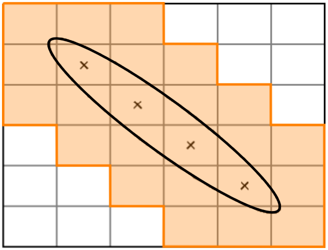

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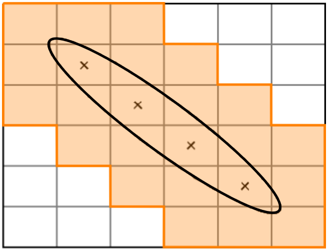

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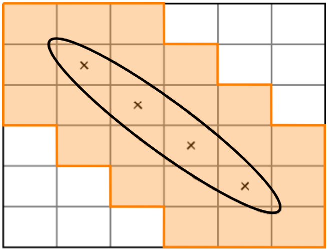

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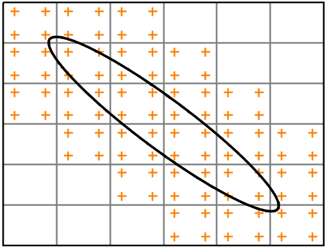
quadrature points



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Minimization over **f**:

$$F(u) = \sum_{Z \subset U_T^{\mathbf{n}}(\Omega)} \left(\int_Z |\nabla u| \, d\mathbf{x} \right) \approx \sum_{\boldsymbol{\theta} \in \Theta} w_{\boldsymbol{\theta}} \|\nabla u(\boldsymbol{\theta})\|_2$$

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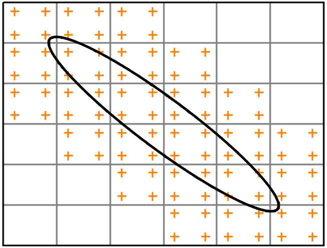

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Tensor product idea

1. inpainting function:

$$u(\mathbf{x}) = \sum_{\mathbf{k}\in K} f_{\mathbf{k}} b_{\mathbf{k}}(\mathbf{x}),\ \mathbf{x} \in R$$

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$$u(\boldsymbol{\xi}) = g(\boldsymbol{\xi})\ ,\ \forall \boldsymbol{\xi} \in \Xi^*$$

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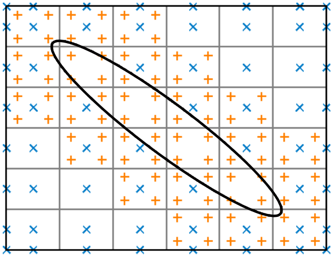
interpolation and quadrature points



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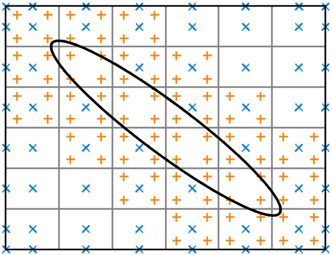

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How to solve this?

Second order cone program

Definition SOCP[1]

Determine $\mathbf{x} \in \mathbb{R}^n$ by

$$\begin{aligned} \text{minimize} \quad & \mathbf{t}^T \mathbf{x} \\ \text{such that} \quad & \|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^T \mathbf{x} + d_i, \quad i = 1, \ldots, N \end{aligned}$$

with parameters $\mathbf{t}, \mathbf{c}_i \in \mathbb{R}^n$, $\mathbf{b}_i \in \mathbb{R}^{n_i - 1}$, $d_i \in \mathbb{R}$ and $\mathbf{A}_i \in \mathbb{R}^{(n_i - 1) \times n}$.

[1] Lobo et al.: "Applications of second-order cone programming", *Linear Algebra and its Applications*, 1998.

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- second order (convex) cone of dimension n_i:

$$\mathcal{K}_{n_i} := \left\{ \left(\begin{smallmatrix}\mathbf{y}\\ s\end{smallmatrix}\right) \middle| \, \mathbf{y} \in \mathbb{R}^{n_i-1}, \, s \in \mathbb{R}, \|\mathbf{y}\|_2 \leq s \right\}$$

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- second order cone constraint of dimension n_i:

$$\|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^T\mathbf{x} + d_i \quad \Longleftrightarrow \quad \begin{pmatrix}\mathbf{A}_i \\ \mathbf{c}_i^T\end{pmatrix}\mathbf{x} + \begin{pmatrix}\mathbf{b}_i \\ d_i\end{pmatrix} \in \mathcal{K}_{n_i}$$

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SOCP for spline inpainting

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SOCP

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Discrete spline inpainting

$$\min_{\mathbf{f}} \quad \sum_{\boldsymbol{\theta}\in\Theta} w_{\boldsymbol{\theta}}\|\nabla u(\boldsymbol{\theta})\|_2$$
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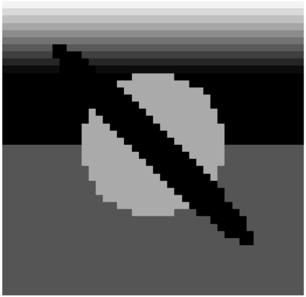
Example: interpolation

Image with gap



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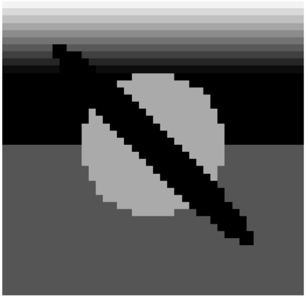
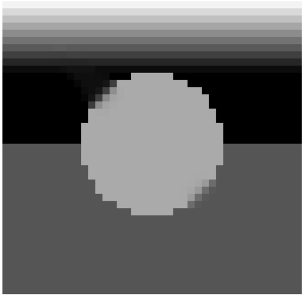
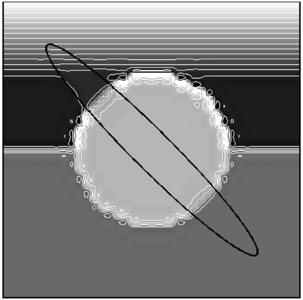


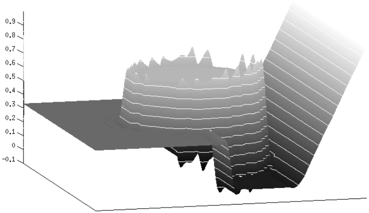
Spline image



Example: interpolation

Spline function





Quasi-interpolants

Quasi-interpolant of order $\boldsymbol{\nu}$

Linear map $Q : \mathcal{C}(R) \to \mathcal{S}_{\mathbf{n}}(T, R)$ such that

$$Qg = \sum_{\mathbf{k}} \lambda_{\mathbf{k}}(g) b_{\mathbf{k}} \text{ and } Qp = p \ \forall p \in \mathbb{P}_{\boldsymbol{\nu}}$$

with uniformly bounded $\lambda_{\mathbf{k}} : \mathcal{C}(R) \to \mathbb{R}$ given by

$$\lambda_{\mathbf{k}}(g) = \sum_{\mathbf{i}(\mathbf{k})} q_{\mathbf{i}(\mathbf{k})}\, g(t_{\mathbf{i}(\mathbf{k})}) \quad \text{for } t_{\mathbf{i}(\mathbf{k})} \in U(\operatorname{supp} b_{\mathbf{k}})$$

Example of order 2: Schoenberg quasi-interpolant

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$$\lambda_{\mathbf{k}}(g) := g(\boldsymbol{\xi}_{\mathbf{k}}), \text{ for Greville abscissae } \boldsymbol{\xi}_{\mathbf{k}} \in R.$$

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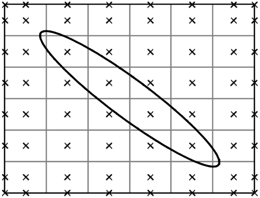
Greville abscissae



order $\mathbf{n} = (3, 3)$

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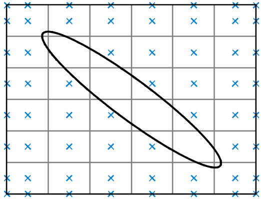
Inpainting problem:

Interpolation points



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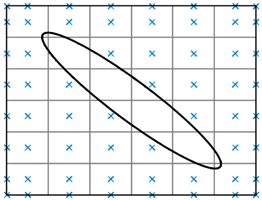

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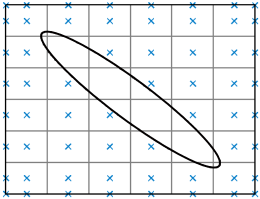

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Quasi-interpolants for spline inpainting

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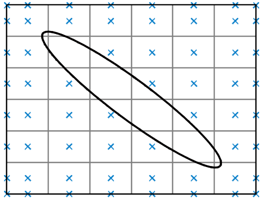

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Quasi-interpolant

$$Q(g) = \sum_{\mathbf{k}} \lambda_{\mathbf{k}}(g) b_{\mathbf{k}}$$

with $\lambda_{\mathbf{k}}(g) := \sum_{\mathbf{i(k)}} q_{\mathbf{i(k)}}\, g(t_{\mathbf{i(k)}})$

Quasi-interpolants for spline inpainting

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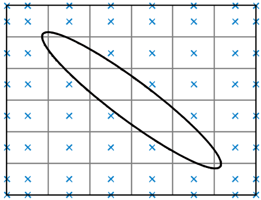

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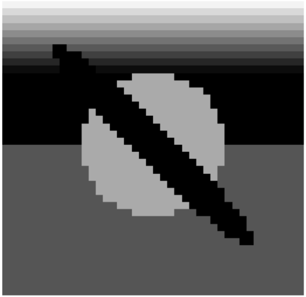
Quasi-interpolant for image g

$$Q(g) = \sum_{\mathbf{j}} g(t_{\mathbf{j}})\tilde{b}_{\mathbf{j}}$$

with $\tilde{b}_{\mathbf{j}} := \sum_{\mathbf{i(j)}} \tilde{q}_{\mathbf{i(j)}}\, b_{\mathbf{i(j)}}$.

Example: quasi-interpolant

Image with gap



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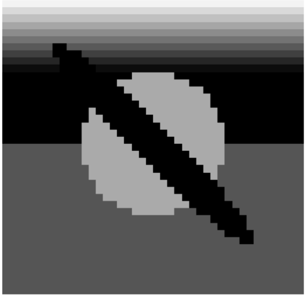
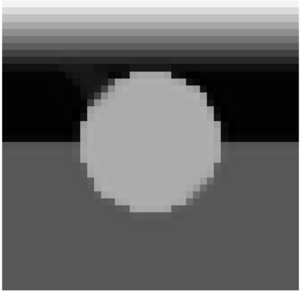


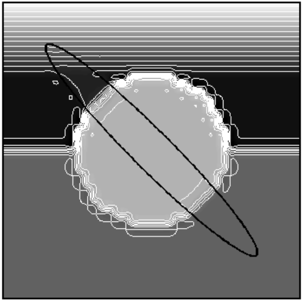
Spline image



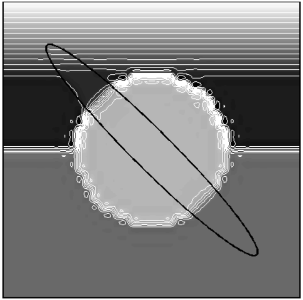
Example: quasi-interpolant

Spline functions

Quasi-interpolant



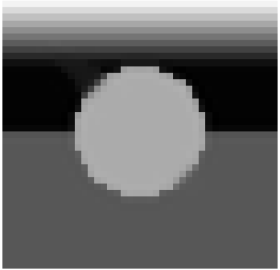
Interpolant



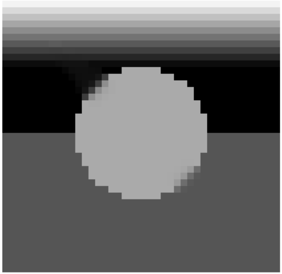
Example: comparison with classical TV

Inpainting results

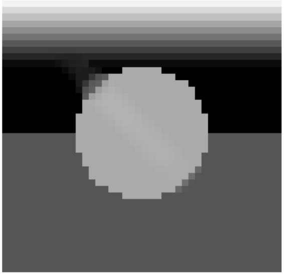
Quasi-interpolant



Interpolant



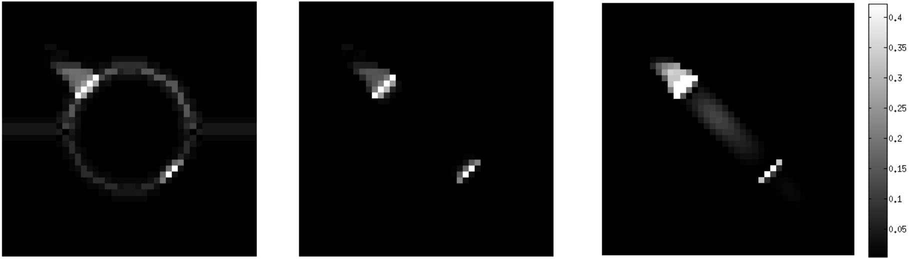
TV



Example: comparison with classical TV

Error of inpainting results

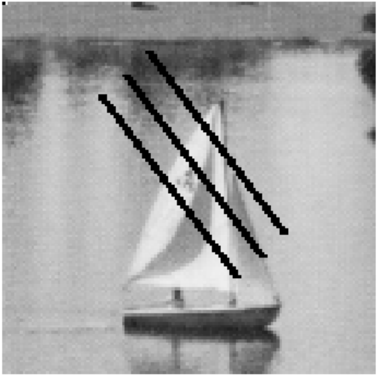
Quasi-interpolant | Interpolant | TV



	max error	mse	$\geq 1/256$	≥ 0.1
Quasi-interpolant	0.4194	$4.9 \cdot 10^{-04}$	318px	10px
Interpolant	0.4229	$4 \cdot 10^{-04}$	57px	9px
TV	$\sim$0.3456	$\sim 5.6 \cdot 10^{-04}$	$\sim$136px	$\sim$17px

Example: sailboat

Image with scratch



Example: sailboat

Inpainting results

Quasi-interpolant

Interpolant

TV

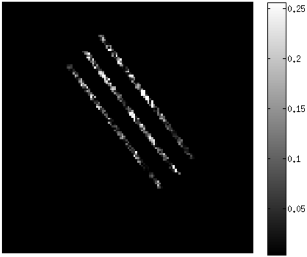
Example: sailboat

Error of inpainting results

Quasi-interpolant

Interpolant

TV



	max error	mse	$\geq 1/256$	≥ 0.1
Quasi-interpolant	0.2257	$7.5 \cdot 10^{-05}$	5962px	13px
Interpolant	0.2171	$5.1 \cdot 10^{-05}$	496px	13px
TV	$\sim$0.2141	$\sim 6.1 \cdot 10^{-05}$	$\sim$516px	$\sim$17px

Summary

Tensor product spline inpainting:

- grid with multiple knots on boundary $\Rightarrow$ stable basis
- (quasi-)interpolation at Greville abscissae $\Rightarrow$ Schoenberg-Whitney
- minimization over union of grid cells $\Rightarrow$ Gauss quadrature
- spline and derivatives with same coefficients $\mathbf{f}$ $\Rightarrow$ optimization w.r.t. $\mathbf{f}$

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Some possible modifications:

- other functionals
- 2 step method for optimization
- iterative solver
- adapt grid or basis

Thank you for your attention!