Scattered Data Problems on (Sub)Manifolds

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Part I: Sparse Scattered Data on (Sub)Manifolds

Problem statement

Setting:

- $\mathbb{M} \subseteq \mathbb{R}^d$ a hypersurface with $q = \dim \mathbb{M} < 4$
  - closed
  - compact
  - without boundary
- $\Xi \subseteq \mathbb{M}$ a set of **sparse** sites scattered over $\mathbb{M}$
- Υ function values to Ξ

Task: Determine a »reasonable« function $f : \mathbb{M} \to \mathbb{R}$ such that

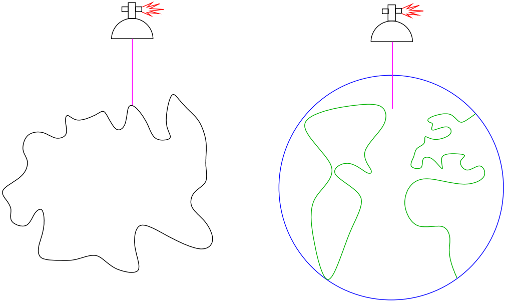
$$f(\xi) = y_\xi \text{ for all } \xi \in \Xi \text{ and corresponding } y_\xi \in \Upsilon$$

Application

Imagine extrapolation of satellite laser measurements on an asteroid or on earth.

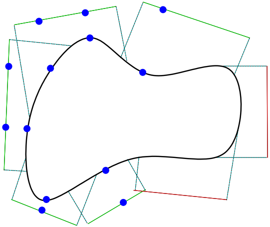
The laser needs time to calibrate (=> only few values on path).

The sattelite can only follow certain distinct orbits.



Common Approaches: Charts and Blending

- Use charts and define function spaces there to solve problem locally.
- Blend local solutions to obtain global solution.



Problems:

- There might be charts without any sites
- Blending tends to produce undesireable »gluing breaks«

Common Approaches: Intrinsic Functions

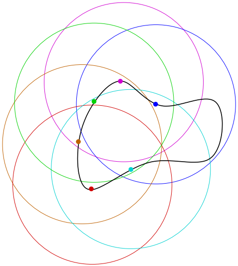
- Determine purely intrinsic function spaces $\mathcal{F}_{\mathbb{M}}$ like spherical harmonics on $\mathbb{S}^q$.
- Use these for a solution

Problems:

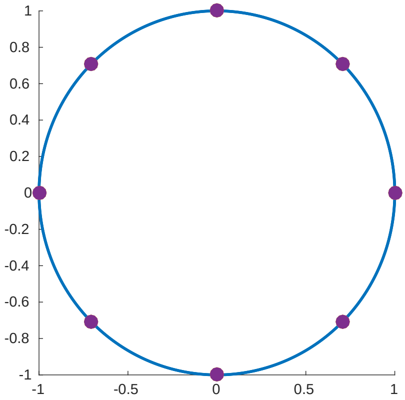
- These spaces are $\mathbb{M}$-specific and for arbitrary $\mathbb{M}$ hard to determine.
- Also, they are often costly to evaluate.

Common Approaches: Extrinsic Direct Interpolation

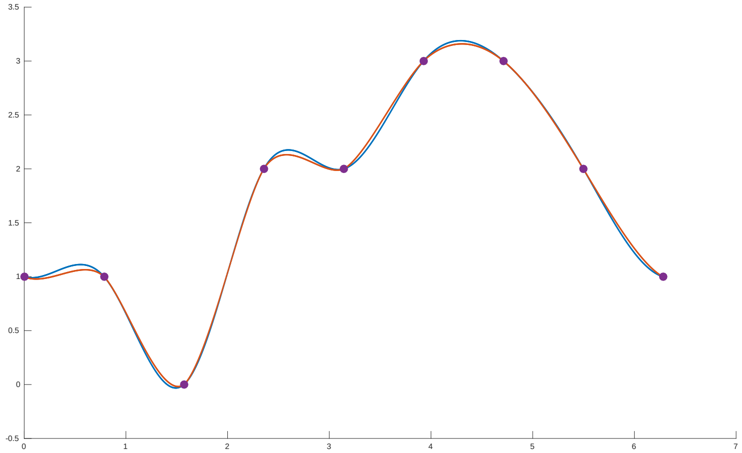
- Use some function space in a suitable ambient neighbourhood of $\mathbb{M}$: RBF, Splines...
- Solve standard interpolation problem in neighbourhood, ignore the geometry of $\mathbb{M}$.
- Restrict the solution to $\mathbb{M}$.



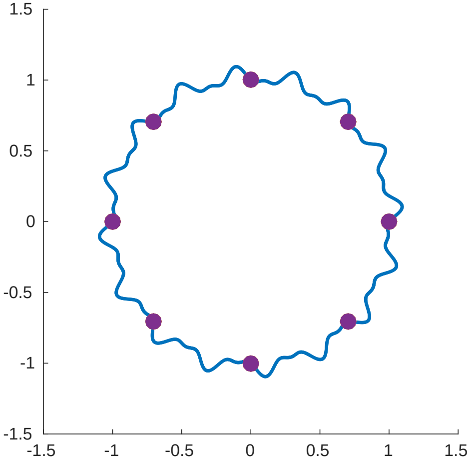
Directly applicable and works always — but how good?



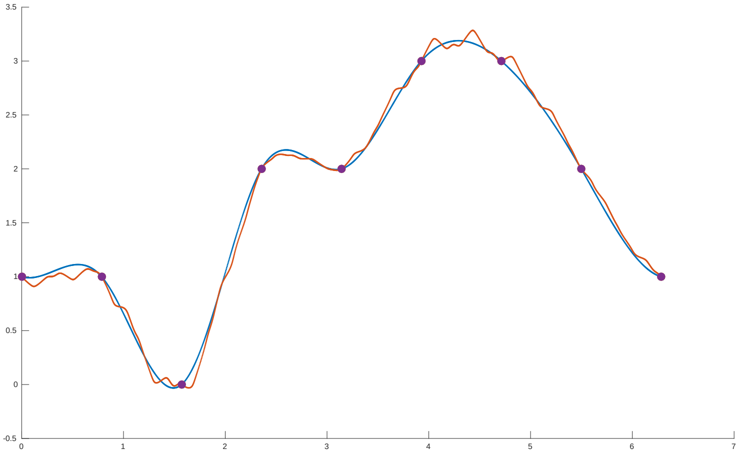
Data sites with function values: 1, 1, 0, 2, 2, 3, 3, 2



Periodic cubic spline | Restricted thin-plate spline | Interpolation sites



Data sites with function values: 1, 1, 0, 2, 2, 3, 3, 2



Periodic cubic spline interpolant

Restricted thin-plate spline interpolant

Common Approaches: Extrinsic Interpolation

- Use some function space in a suitable ambient neighbourhood of $\mathbb{M}$: RBF, Splines...
- Solve standard interpolation problem in neighbourhood, ignore the geometry of $\mathbb{M}$.
- Restrict the solution to $\mathbb{M}$.

Problems:

- Difficulties occur for sparse data.
- Suffers extremely from *intricate* geometries.

New Approach

- Use some function space in a suitable ambient neighbourhood of $\mathbb{M}$: RBF, Splines...
- Transfer intrinsic properties into extrinsic (ambient) properties — approximately.
- Solve approximately intrinsic problem with extrinsic methods.

Pro's:

- Extrinsic function spaces are well understood.
- Extrinsic function space are applicable to any submanifold.
- Easily understood and implemented even for non-mathematicians.

Solution idea: Background

DEFINITION (Tangential Derivative)

Let $f : \mathbb{M} \to \mathbb{R}$ be a sufficiently (weakly) differentiable function and $\tilde{f}$ an extension into $U(\mathbb{M})$. The **Tangential Derivative Operator $\mathbf{d}_{\mathbb{M}}$** is defined as

$$\mathbf{d}_{\mathbb{M}} f := \mathbf{d}\tilde{f} - \pi_N(\mathbf{d}\tilde{f})$$

and independent of the choice of $\tilde{f}$.

Solution idea: Background

THEOREM

Let $f \in C^2(\mathbb{M})$, and $\overline{f}$ an extension that is **constant in normal directions** of $\mathbb{M}$.

Then the 1^{st} and 2^{nd} **tangential** derivatives of f coincide with the **euclidean** 1^{st} and 2^{nd} derivatives of $\overline{f}$ on the tangent space (e.g. [Dziuk/Elliot, 2013]).

Solution idea: Background

What does that mean?

- Euclidean derivatives of $\bar{f}$ give access to tangential derivatives of f
- Euclidean methods can be used to handle intrinsic problems
- Intrinsic functionals are easily transferred into Euclidean setting
- Also: Laplacian of $\bar{f}$ allows access to Laplace-Beltrami of f

Solution idea: Background

THEOREM (M. 2015)

Let $f \in C^2(\mathbb{M})$, and $\tilde{f}$ an arbitrary extension. Then the deviations of the 1^{st} and 2^{nd} **tangential** derivatives of f from the **euclidean** 1^{st} and 2^{nd} derivatives of $\tilde{f}$ on the tangent space are **Lipschitz functions** of the first normal derivatives.

Solution idea: Background

What does that mean?

- Euclidean derivatives of $\tilde{f}$ give **approximate** access to tangential derivatives of f
- Standard methods can be used to handle intrinsic problems **approximately**
- Intrinsic functionals are easily **approximated** by standard functionals

New Approach: Solution Idea

Let $U(\mathbb{M})$ be a neighbourhood of $\mathbb{M}$ s.t. any $x \in U(\mathbb{M})$ has a unique closest point on $\mathbb{M}$.

Consider a suitable function space $\mathcal{F}(U(\mathbb{M}))$ on $U(\mathbb{M})$.

Minimize squared 2^{nd} derivative in tangent directions over $\mathbb{M}$

such that:

Interpolation in Ξ holds

Normal derivatives $\to 0$

Optimization Functional: Tensor-Product B-Splines

Minimize for grid width $h > 0$ and $\tau_1, ...\tau_q$ an ONB of $T_p(\mathbb{M})$

$$\mathbf{E}_{\mathbb{M}}(s_h) := \int_{\mathbb{M}} \sum_{i,j=1}^{q} \left| \frac{\partial^2}{\partial \tau_i \partial \tau_j} s_h \right|^2$$

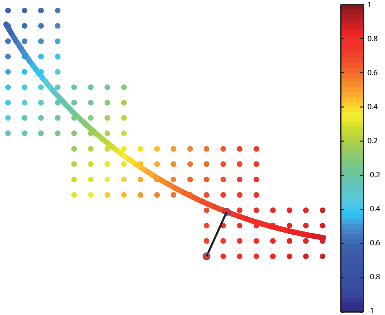
such that

$$s_h(\xi) = y_\xi \quad \forall \xi \in \Xi$$

and for any normal direction ν as $h \to 0$

$$\left| \frac{\partial s_h}{\partial \nu} \right| \leq h^\alpha$$

with $\alpha > 1$.



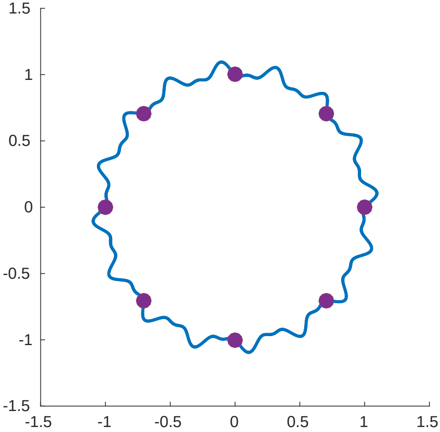
A grid region around a smooth submanifold part — function values extended constantly along normals.

Theory: Solvability and Convergence

THEOREM (M. 2015):

1. For TP-B-Splines and sufficiently small h, the above problem is uniquely solvable.
2. For $h \to 0$ the energy $\mathbf{E}_{\mathbb{M}}(s_h)$ converges to the unique optimal energy in $\mathcal{H}^2(\mathbb{M})$.
3. Restrictions of optimal splines $s_h|_{\mathbb{M}}$ approach unique optimum $f^* \in \mathcal{H}^2(\mathbb{M})$:

$$\|s_h - f^*\|_{\mathcal{H}^2(\mathbb{M})} \to 0 \quad \text{as} \quad h \to 0.$$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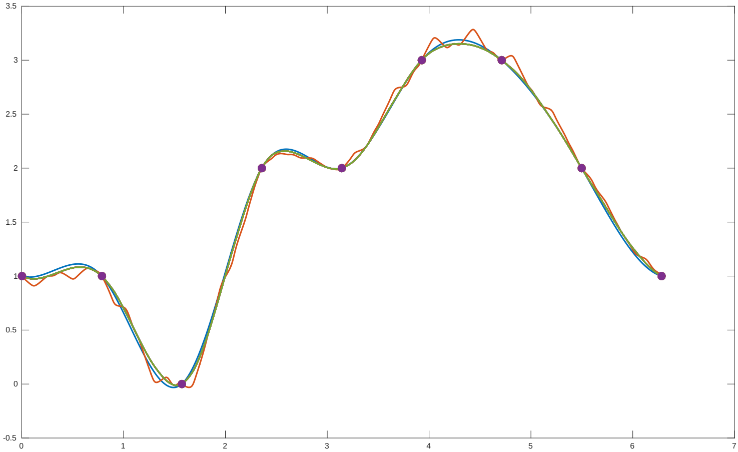
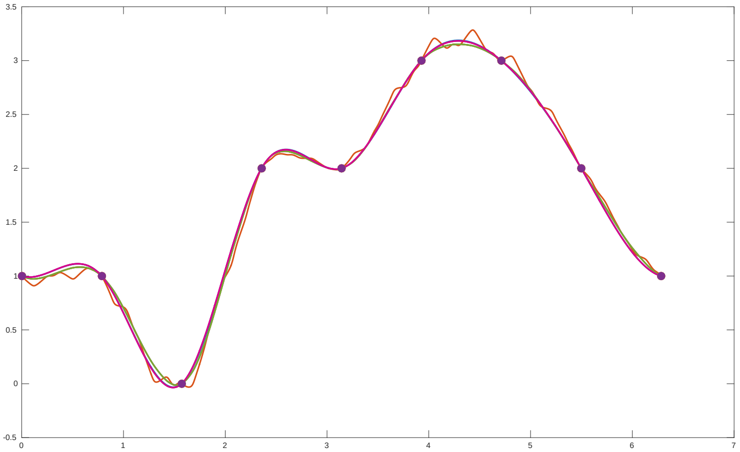


Periodic cubic spline interpolant

Restricted thin-plate spline interpolant

Our optimum for $h = 0.02$

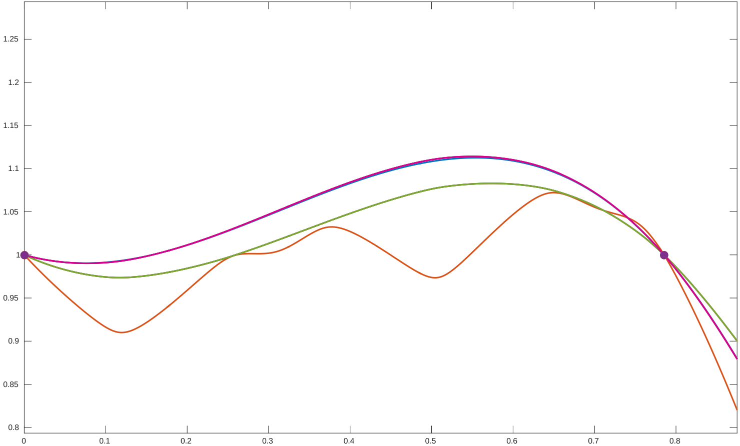


Periodic cubic spline interpolant

Opt. for $h = 0.02$ ($err_{RMS} \approx 10^{-2}$)

Restricted thin-plate spline interpolant

Opt. for $h = 0.01$ ($err_{RMS} \approx 10^{-3}$)

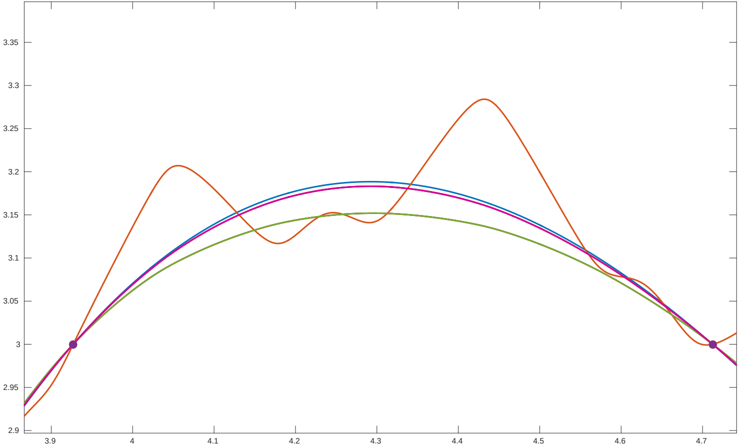


Periodic cubic spline interpolant

Restricted thin-plate spline interpolant

Opt. for $h = 0.02$ ($err_{RMS} \approx 10^{-2}$)

Opt. for $h = 0.01$ ($err_{RMS} \approx 10^{-3}$)

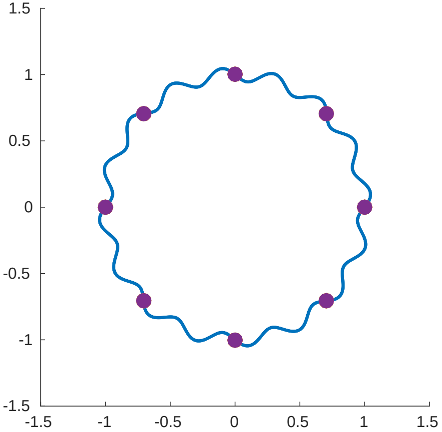


Periodic cubic spline interpolant

Restricted thin-plate spline interpolant

Opt. for $h = 0.02$ ($err_{RMS} \approx 10^{-2}$)

Opt. for $h = 0.01$ ($err_{RMS} \approx 10^{-3}$)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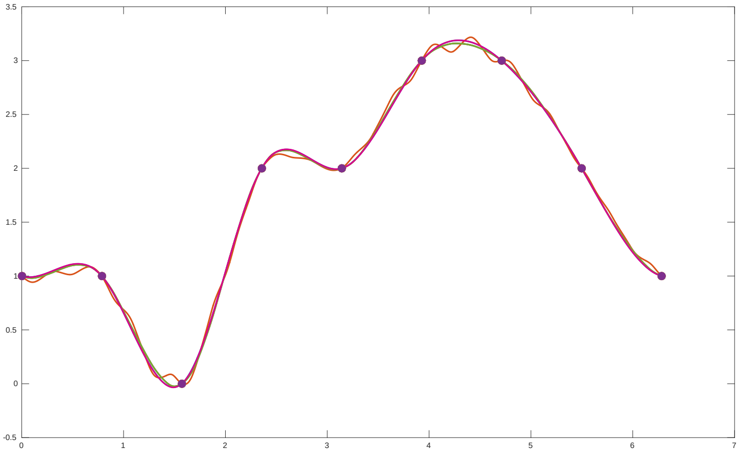
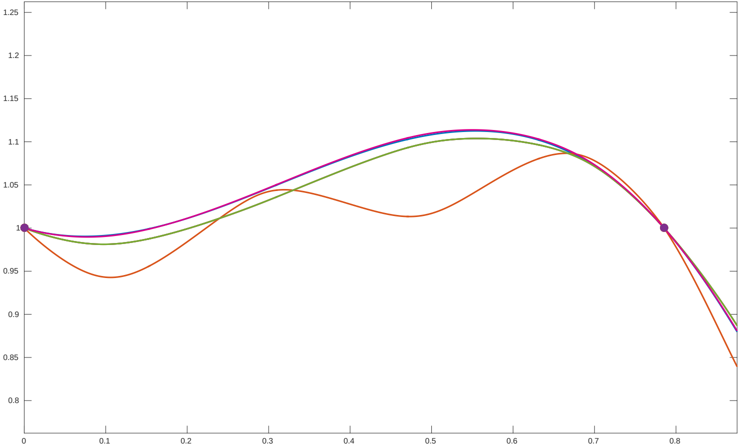


Periodic cubic spline interpolant

Opt. for $h = 0.05$ ($err_{RMS} \approx 10^{-2}$)

Restricted thin-plate spline interpolant

Opt. for $h = 0.025$ ($err_{RMS} \approx 10^{-3}$)

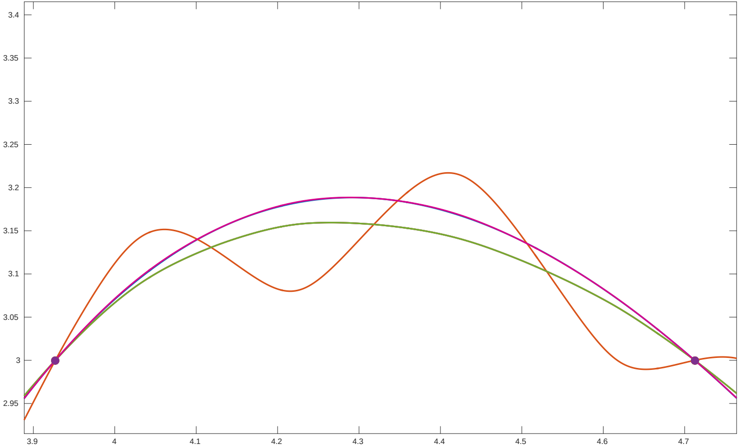


Periodic cubic spline interpolant

Opt. for $h = 0.05$ ($err_{RMS} \approx 10^{-2}$)

Restricted thin-plate spline interpolant

Opt. for $h = 0.025$ ($err_{RMS} \approx 10^{-3}$)

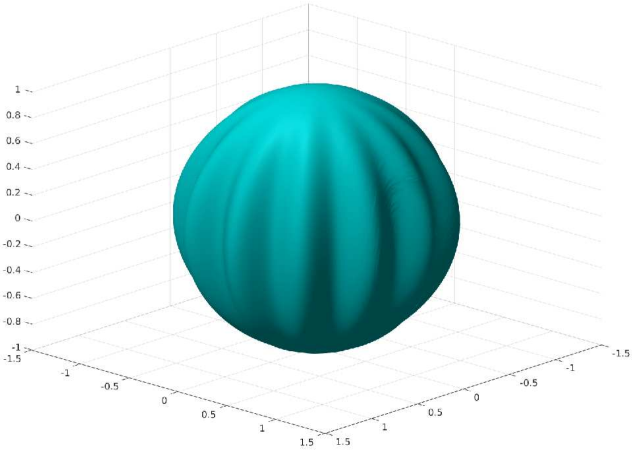


Periodic cubic spline interpolant

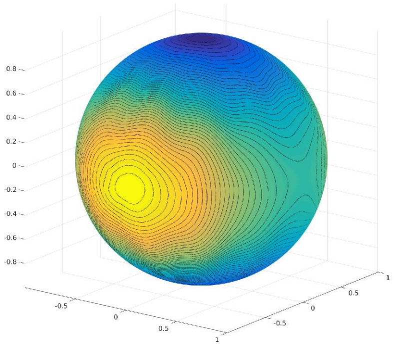
Opt. for $h = 0.05$ ($err_{RMS} \approx 10^{-2}$)

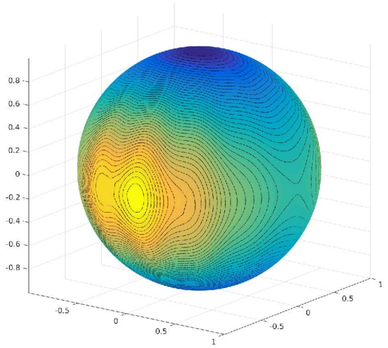
Restricted thin-plate spline interpolant

Opt. for $h = 0.025$ ($err_{RMS} \approx 10^{-3}$)



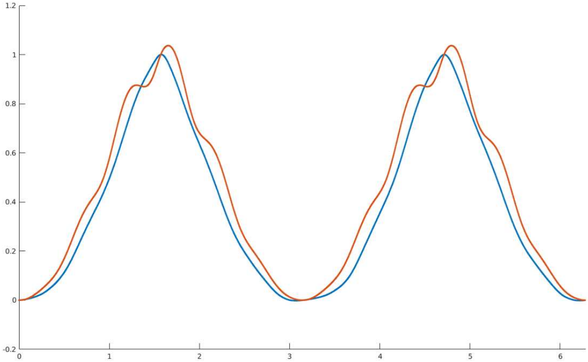
Sites with values: $s(\pm e_1) = 0,\ s(\pm e_2) = 1,\ s(\pm e_3) = -1$





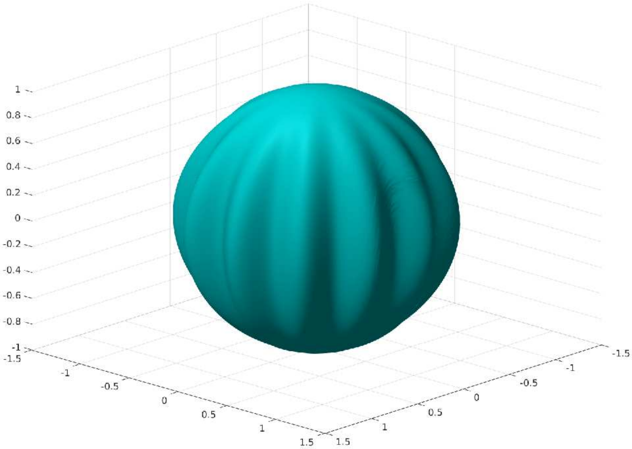
Left: $\mathbb{S}^2$-reproj. Opt. for $h = 0.075$

Right: $\mathbb{S}^2$-reproj. thin-plate interpol.

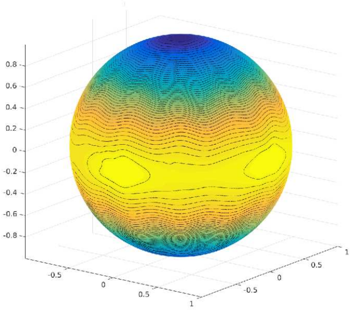


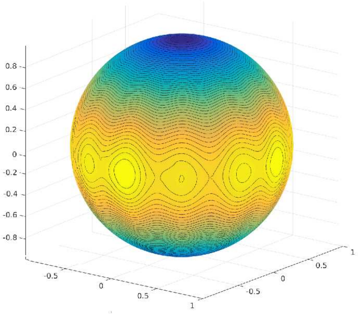
Equator eval. of our opt. ($h = 0.075$)

Equator eval. of thin-plate interpolation



Sites with values: $s(\pm e_1) = s(\pm e_2) = 1,\ s(e_3) = s(-e_3) = -1$

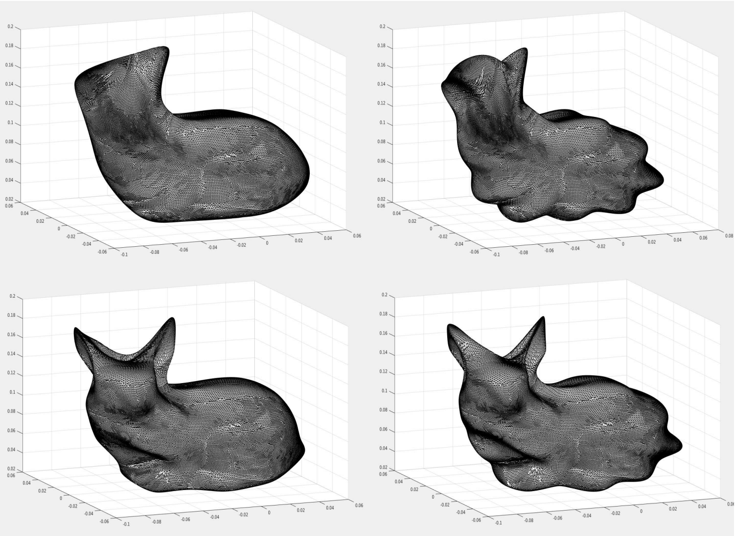




Left: $\mathbb{S}^2$-reproj. Opt. for $h = 0.0625$

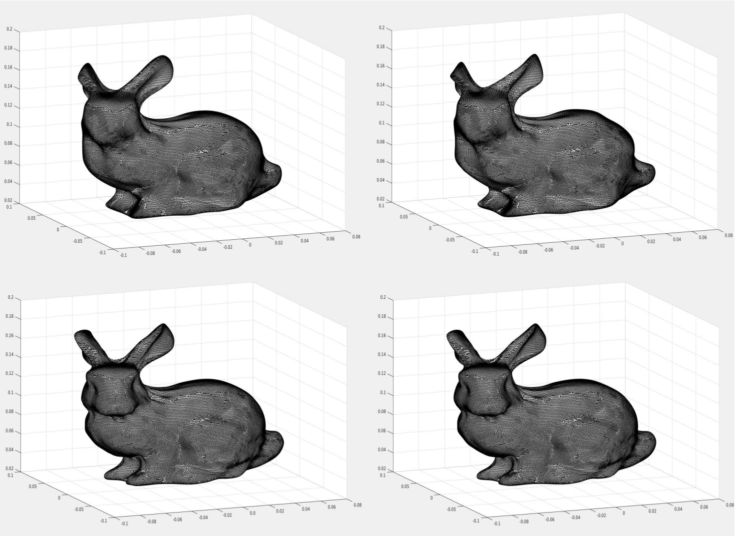
Right: $\mathbb{S}^2$-reproj. thin-plate interpol.

Stanford Bunny interpolation on »pumpkin« in 24 (upper row) and 82 (lower row) sites.



Left column: new optimization result *Right column*: thin-plate interpolation result

Stanford Bunny interpolation in 179 (upper row) and 676 (lower row) sites.



Left column: new optimization result *Right column*: thin-plate interpolation result

Further remarks

► With fill distance $h_{\Xi,\mathbb{M}} = \max_{p\in\mathbb{M}} \min_{\xi\in\Xi} \|p - \xi\|_2$ decreasing, the convergence is about that of thin-plate splines

But: One will have to increase the number of DOF correspondingly to meet

- More interpolation conditions
- Sufficient constance in normal directions

Part II: Smoothing on (Sub)Manifolds

Problem statement

Setting:

- $\mathbb{M} \subseteq \mathbb{R}^d$ a hypersurface with $\dim\mathbb{M} < 4$
  - closed
  - compact
  - without boundary
- $\Xi \subseteq \mathbb{M}$ a set of data sites scattered over $\mathbb{M}$
- Υ function values to Ξ, **possibly noisy**

Task: Determine a »reasonable« function $f : \mathbb{M} \to \mathbb{R}$ such that

$f(\xi)$ approximate y_ξ for all $\xi \in \Xi$ and corresponding $y_\xi \in \Upsilon$ **w.r.t. smoothness of** f

Common Approaches: Problems Revisited

Chart&Blending Problems:

- There might be charts without any sites
- Blending tends to produce undesireable »gluing breaks«
- **What does smoothness in a chart mean for smoothness on $\mathbb{M}$?**

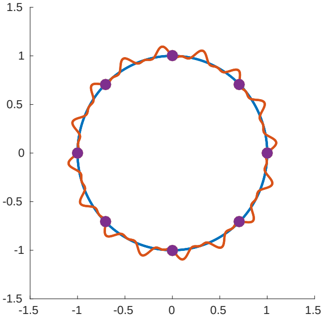
Intrinsic Space Problems:

- These spaces are $\mathbb{M}$-specific and for arbitrary $\mathbb{M}$ hard to determine.
- Also, they are often costly to evaluate. • **Accessing »smoothness« and intrinsic derivatives is complicated**.

Direct Extrinsic Smoothing Problems:

- Difficulties occur for sparse data.
- Suffers extremely from *difficult* geometries.
- The smoothest function w.r.t. 2^{nd} derivatives is a least squares linear polynomial.

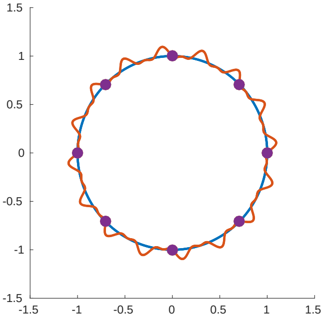
  What does that mean on $\mathbb{M}$?



Direct Extrinsic Smoothing Problems:

- Difficulties occur for sparse data.
- Suffers extremely from *difficult* geometries.
- The smoothest function w.r.t. 2^{nd} derivatives is a least squares linear polynomial.

What does that mean on $\mathbb{M}$? **Nothing!**



Direct Extrinsic Smoothing Problems:

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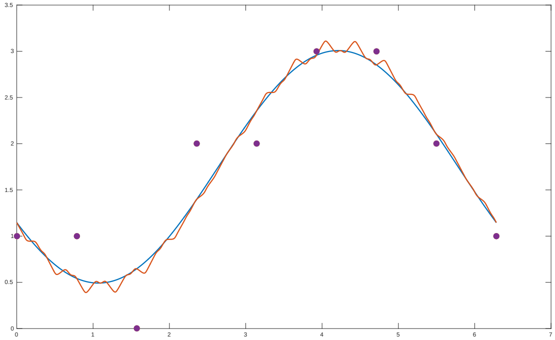


Linear Least Squares Polynomial, evaluated on the submanifolds

New Approach: Solution Idea

Let $U(\mathbb{M})$ be a neighbourhood of $\mathbb{M}$ s.t. any $x \in U(\mathbb{M})$ has a unique closest point on $\mathbb{M}$.

Consider a suitable function space $\mathcal{F}(U(\mathbb{M}))$ on $U(\mathbb{M})$.

Convex weighted Minimization

of

Squared 2^{nd} derivative in tangent directions on $\mathbb{M}$

versus

Discrete approximation error in Ξ

such that:

Normal derivatives $\to 0$

Optimization Functional: Tensor-Product B-Splines

Minimize for grid width $h > 0$, $\tau_1, ...\tau_q$ an ONB of $T_p(\mathbb{M})$, $\eta \in]0,1[$, $\eta_\xi \in]0,1] \forall \xi \in \Xi$

$$\mathbf{E}_{\mathbb{M},\eta}(s_h) := \eta \cdot \int_{\mathbb{M}} \sum_{i,j=1}^{q} \left| \frac{\partial^2}{\partial \tau_i \partial \tau_j} s_h \right|^2 + (1-\eta) \sum_{\xi \in \Xi} \eta_\xi \cdot (s_h(\xi) - y_\xi)^2$$

such that for any normal direction ν as $h \to 0$

$$\left| \frac{\partial s_h}{\partial \nu} \right| \leq h^\alpha$$

with $\alpha > 1$.

Further Remarks

- η balances smoothness and discrete approximation (as usual)
- The $\{\eta_\xi\}_{\xi\in\Xi}$ can be used to balance data density
- Additional strict interpolation conditions are possible

Theory: Solvability and Convergence

THEOREM (M. 2015):

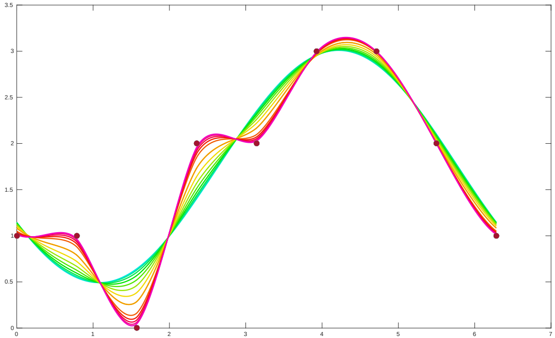
1. For TP-B-Splines and sufficiently small h, the above problem is uniquely solvable.

2. For $h \to 0$ the energy $\mathbf{E}_{\mathbb{M},\eta}(s_h)$ converges to the unique optimal balanced energy in $\mathcal{H}^2(\mathbb{M})$.

3. Restrictions of optimal splines $s_h|_{\mathbb{M}}$ approach unique optimum $f_\eta^* \in \mathcal{H}^2(\mathbb{M})$:

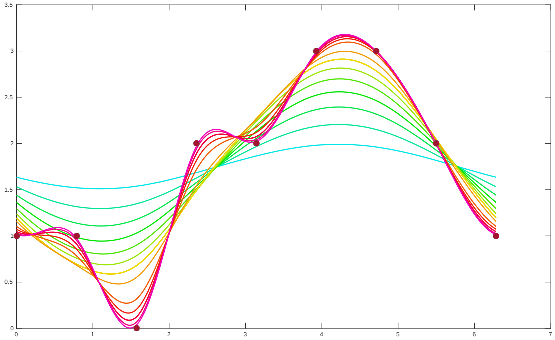
$$\left\| s_h - f_\eta^* \right\|_{\mathcal{H}^2(\mathbb{M})} \to \quad 0 \text{ as } \quad h \to 0.$$

Direct extrinsic smoothing on $\mathbb{S}^1$ for different weights with thin-plate splines:



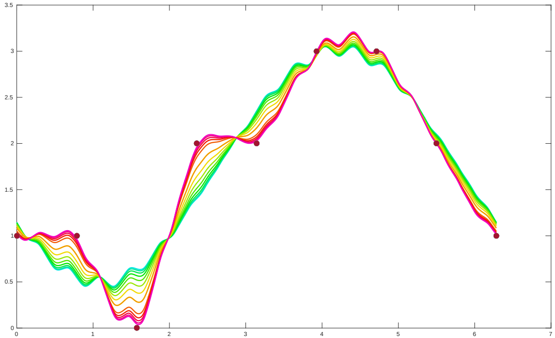
There is an obvious non-constant minimal second tangential derivative square integral!

Approximate intrinsic smoothing on $\mathbb{S}^1$ for different weights with $h = 0.05$:



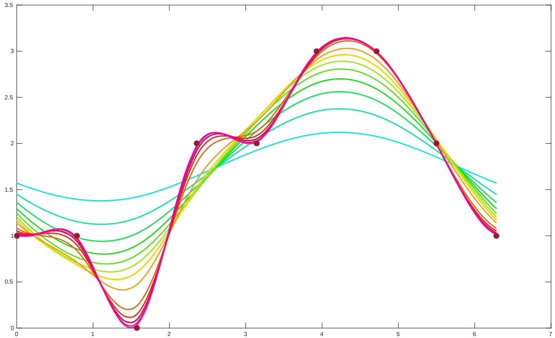
Square tangential derivative integrals can achieve arbitrary small values!

Direct extrinsic smoothing on »Flower« for different weights with thin-plate splines:

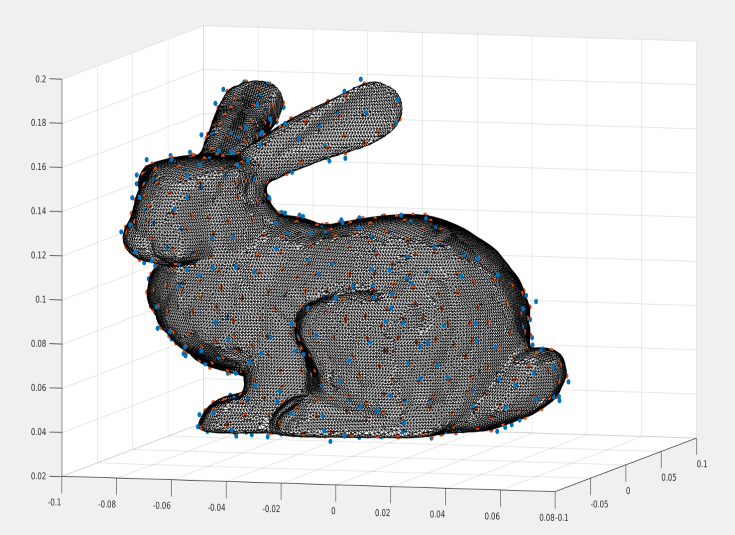


Geometric features of $\mathbb{M}$ are reproduced and ruin the smoothness!

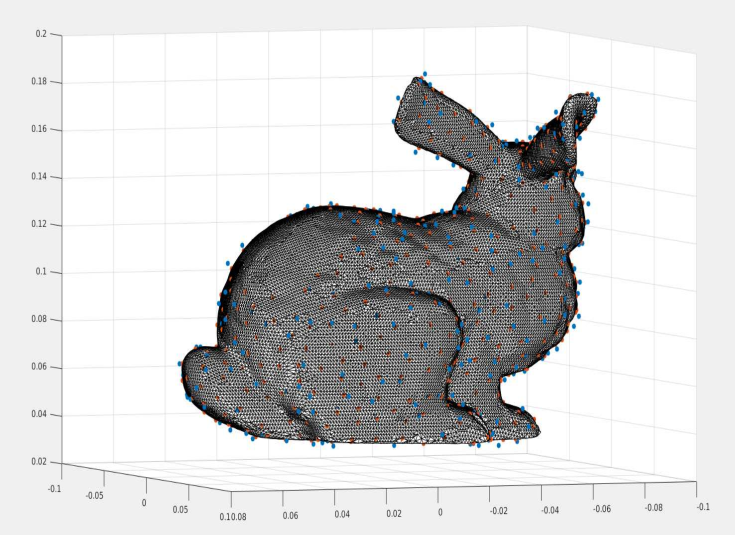
Approximate intrinsic smoothing on »Flower« for different weights with $h = 0.02$:



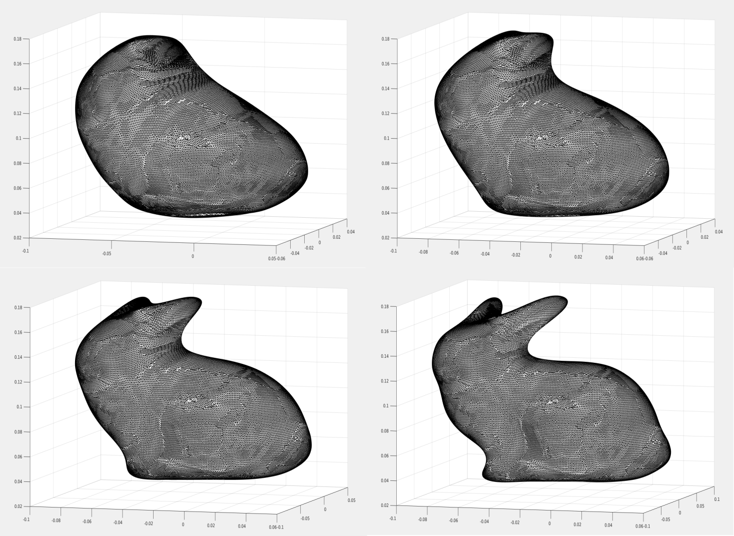
Geometric features of $\mathbb{M}$ play no relevant role!



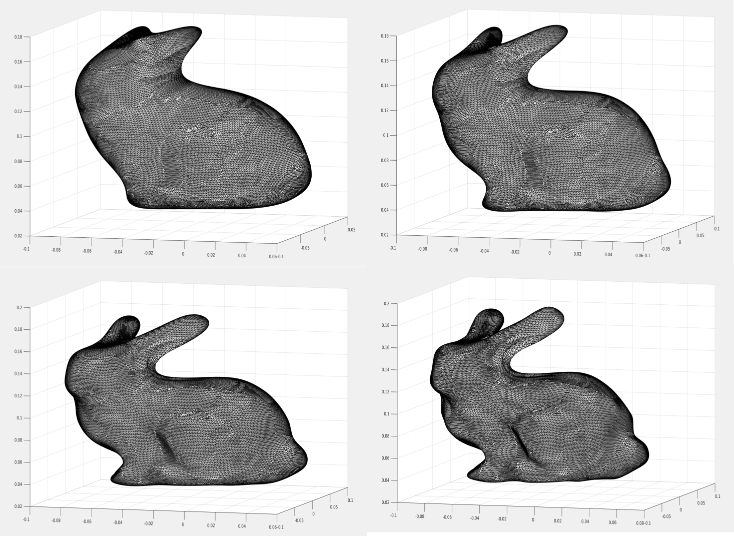
Noisy and irregular sampling of Stanford Bunny: Front view.



Noisy and irregular sampling of Stanford Bunny: Back view.

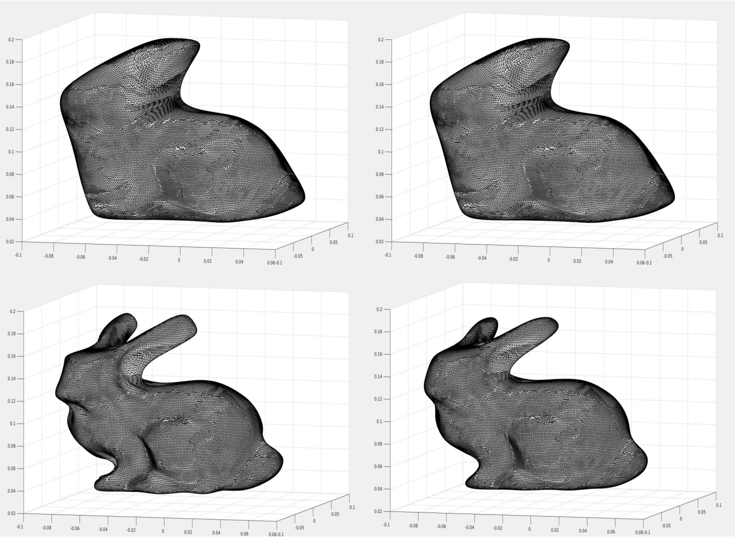


Outcome of Bunny Smoothing on »Pumpkin« for different smoothing weights.



Outcome of Bunny Smoothing on »Pumpkin« for different smoothing weights.

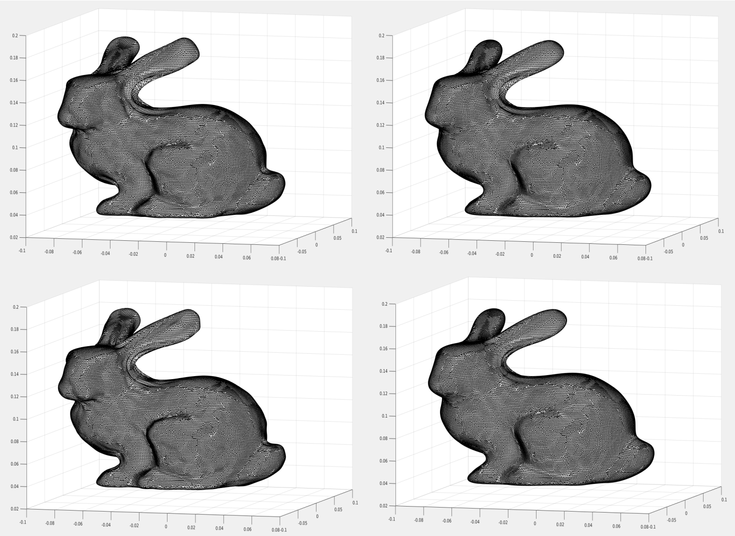
Bunny approximation on »Pumpkin« in 38 (upper row) and 353 (lower row) sites.



Left column: Interpolation

Right column: Slight Smoothing

Bunny approximation on »Pumpkin« in 751 (upper row) and 3781 (lower row) sites.



Left column: Interpolation

Right column: Slight Smoothing

Summary

New approaches to sparse scattered data interpolation and discrete data smoothing on submanifolds:

- Overcome common difficulties
- Apply well-known concepts in novel setting
- Easy to implement
- Produce pleasant results