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# Riesz based multiresolution analysis in image processing

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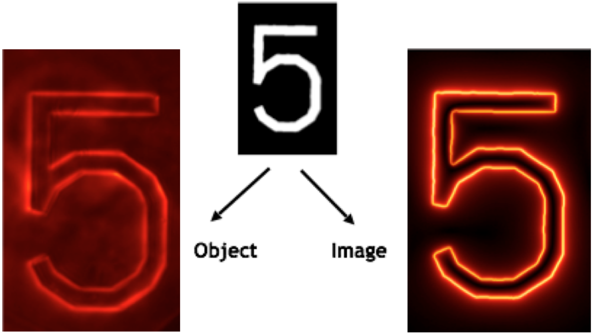
# MOTIVATION

Illustration: Edge enhancement by spiral phase filtering vs. monogenic signal approach




Optical Imaging Fourier plane filtering: Spiral phase filter

Image Processing Fourier Domain: Riesz/Radial Hilbert Transform

# HILBERT TRANSFORM – ANALYTIC SIGNAL

- Many natural and man-made signals exhibit time-varying frequencies (e.g., chirps, images, radio waves).
- Characterization and analysis of such signals $f(t)$, based on
  - *instantaneous amplitude* $a(t)$,
  - *instantaneous phase* $\varphi(t)$,
  - *instantaneous frequency* $\omega(t) := \varphi'(t)$,

$$f(t) = a(t)\cos(\varphi(t)).$$

- It is convinient to use a complexified version of the signal whose real part is a given real-valued signal $f(t)$.
  There are infinitely many ways to define a $f(t) + ig(t)$.

# ANALYTIC SIGNAL

- Gabor (1946) proposed to use the *Hilbert transform* of $f(t)$ as $g(t)$, and called the complex-valued $f(t) + i(\mathcal{H}f)(t)$ an *analytic signal*.
- Vakman (1972) proved that $g(t)$ must be of the Hilbert transform of $f(t)$ if we impose some a priori physical assumptions:
  - $g(t)$ must be derived from $f(t)$.
  - Amplitude continuity: a small change in $f(t)$ leds to a small change in $a(t)$.
  - Phase independence of scale: if $cf(t),\ c \in \mathbb{R}$, arbitrary scalr, then the phase does not change from that of $f(t)$ and its amplitude bacomes $c$ times that of $f(t)$.
  - Harmonic correspondence; if $f(t) = a_0 \cos(\omega_0 t + \varphi_0)$, then $a(t) = a_0$ and $\varphi(t) = \omega_0 t + \varphi_0$.

# MATHEMATICALLY DESCRIPTION

Real part $f(t) = a(t)\cos(\varphi(t))$,
then $\mathcal{H}(a(t)\cos(\varphi(t))) = a(t)\mathcal{H}(\cos(\varphi(t)))$ *Bedrosian identity*
$a(t)\mathcal{H}(\cos(\varphi(t))) = a(t)\sin(\varphi(t))$ for $a(t) = a_0$ and $\varphi(t) = \omega_0 t + \varphi_0$.

Using complex numbers: $f(t) + i(\mathcal{H}f)(t) = a(t)\exp(\varphi(t))$.
*Generalizations into higher dimensions?*

- Analytic signal = boundary values of analytic functions in $\mathbb{C}^2$. (Hahn)
- Spiral phase quadrature formula (Larkin)
- Monogenic signal (Felsberg, Sommer)

Using complex numbers, vectors, hypercomplex numbers, quaternions, Clifford algebras:

- Riesz transforms $\mathcal{R}_j f(\vec{x}) = \frac{2}{A_n}\int_{\mathbb{R}^n} \frac{x_j - u_j}{|\underline{x}-\underline{u}|^{n+1}} f(\underline{u})\, d\underline{u}$.
- $\mathcal{H} = \sum_{j=1}^{n} e_j \mathcal{R}_j$ Hilbert-Riesz transform, $e_i e_j + e_j e_i = -2\delta_{ij}$.
- Monogenic signal $(I + \mathcal{H})f(t)$ boundary values of null solution of a Dirac operator.

# MONOGENIC SIGNAL

The monogenic signal are therefore the boundary values of an monogenic function in the upper half space and can be identfied with a quaternion:

$$f_M = f + e_1 R_1 f + e_2 R_2 f = Ae^{\underline{u}\varphi},$$

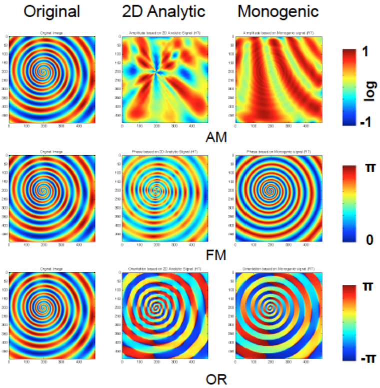
where

- $A(x_1,\ x_2)$ is the amplitude,
- $\underline{u}(x_1, x_2)$ the orientation (vector) and
- $\varphi(x_1, x_2)$ is the phase.

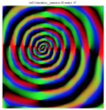
The monogenic signal will be compared with another generalization the 2D analytic signal in higher dimensions which is based on several complex variables and the partial and total Hilbert transforms in $\mathbb{C}^2$.

# EXAMPLE 1 – THEORETICAL EXAMPLE

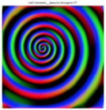
Synthetic Data: 2D Amplitude-Frequency-modulated Signal
(2D AM-FM signal)




Orientation in RGB representation



2D Analytic

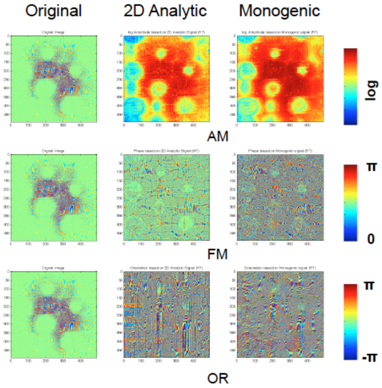


Monogenic

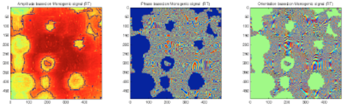
# EXAMPLE 2 – REAL LIFE

Real World Data: Interferometric fringe pattern

recorded by Full-Field Optical Coherence Tomography (FF-OCT) Imaging




Amplitude value as confidence mask for phase and orientation



# RIESZ-HILBERT TRANSFORM

- Hilbert transform $\mathcal{H}f(\underline{x}) = \sum_{j=1}^{n} e_j \mathcal{R}_j f(\underline{x}) \sim (\mathcal{R}_1 f(\underline{x}),\, \mathcal{R}_2 f(\underline{x}),\, \ldots,\, \mathcal{R}_n f(\underline{x}))^T$, where $R_j$ are the j-th Riesz transform and with Fourier transform $\hat{h} = \frac{i\underline{\omega}}{|\underline{\omega}|}$.
- Directional Hilbert transform $\langle \underline{u}, \mathcal{H}_{\underline{u}} f(\underline{x})\rangle$, where $\underline{u}$ is a unit vector, lead to a Hilbert-like behavior in direction $\underline{u}$: $\widehat{\mathcal{H}_{\underline{u}}}(\underline{\omega})\Big|_{\underline{\omega}=\omega\underline{u}} = -i\,\mathrm{sgn}\,(\omega)$.
- Shift invariance $\forall \underline{x}_0 \in \mathbb{R}^n \quad \mathcal{H}\{f(\cdot - \underline{x}_0)\}(\underline{x}) = \mathcal{H}\{f(\cdot)\}(\underline{x} - \underline{x}_0)$.
- Scale invariance $\forall a \in \mathbb{R}_+ \quad \mathcal{H}\{f(\cdot/a)\}(\underline{x}) = \mathcal{H}\{f(\cdot)\}(\underline{x}/a)$.
- Maps wavelets into gradient-like wavelets

$$\mathcal{H}\left\{\psi\left(\frac{\cdot - \underline{x}_0}{a}\right)\right\}(\underline{x}) = \nabla\{\varphi\}\left(\frac{\cdot - \underline{x}_0}{a}\right),\ \varphi = \mathcal{F}^{-1}\left\{i\frac{\hat{\varphi}(\underline{\omega})}{|\underline{\omega}|}\right\}$$

- Self-reversibility $\forall f \in L^2(\mathbb{R}^n) \quad \mathcal{H}^*\mathcal{H}f(\underline{x}) = \sum_{j=1}^{n} \mathcal{R}_j^* \mathcal{R}_j f(\underline{x}) = f(\underline{x})$

# WHAT KIND OF WAVELET-FRAME ?

Frequency domain design of bandlimited wavelets

## Theorem

*Let $h(\omega)$ be a radial frequency profile such that*

1. $h(\omega) = 0 \quad \forall \omega > \pi$ *(Bandlimited)*
2. $\sum_{l \in \mathbb{Z}} |h(2^l \omega)|^2 = 1$ *(Self-resersibility)*
3. $\left. \frac{d^n h(\omega)}{d\omega^n} \right|_{\omega=0} = 0, \; n = 0, 1, \ldots, N$ *(Vanishing moments)*

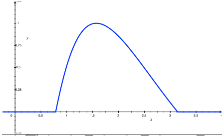
*Then the isotropic mother wavelet $\psi$ with $\hat{\psi}(\underline{\omega}) = h(|\underline{\omega}|)$ generates a tight wavelet frame of $L^2(\mathbb{R}^n)$ whose basis functions*

$$\psi_{i,\underline{k}}(\underline{x}) = \psi_i(\underline{x} - 2\underline{k}) \quad \textit{with} \quad \psi_i(\underline{x}) = 2^{-ni}\psi(2^{-i}\underline{x})$$

*are isotropic with vanishing moments up to order N.*

Example: Simoncelli's wavelets

$$h(\omega) = \begin{cases} \cos\left(\frac{\pi}{2}\log_2\left(\frac{2\omega}{\pi}\right)\right), & \frac{\pi}{4} < |\omega| \leq \pi, \\ 0, & \textit{otherwise} \end{cases}$$



# CONSTRUCTION

Primary tight wavelet frame of $L^2(\mathbb{R}^n)$

$$\forall f \in L^2(\mathbb{R}^n) \quad f(\underline{x}) = \sum_{l\in\mathbb{Z}} \sum_{\underline{k}\in\mathbb{Z}^m} \langle f,\, \psi_{l,\underline{k}}\rangle_{L^2} \psi_{l,\underline{k}}(\underline{x})$$

and $\psi_{l,\underline{k}}(\underline{x}) = 2^{-\frac{lm}{2}}\, \psi\left(2^{-l}(\underline{x} - 2^l \underline{k})\right)$.

Then, $\{\mathcal{H}\psi_{l,\underline{k}} = \nabla\phi_{l,\underline{k}}\}$ is a tight frame such that

$$\forall f \in L^2(\mathbb{R}^n) \quad f(\underline{x}) = \sum_{l\in\mathbb{Z}} \sum_{\underline{k}\in\mathbb{Z}^m} \underline{w}_{l,\underline{k}}\, \mathcal{H}\psi_{l,\underline{k}}(\underline{x}), \quad \underline{w}_{l,\underline{k}} = \langle f, \mathcal{H}\psi_{l,\underline{k}}\rangle$$

It also consists a monogenic frame when $\mathcal{H}$ is replaces by $(I + \mathcal{H})$.

# WHAT DO WE GET?

**Wavelet coefficients**: $w_l[k] = \langle f, \psi_{l,k} \rangle$, $\underline{w}_l[k] = \langle f, \mathcal{H}\psi_{l,k} \rangle$

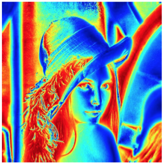
For a 2D image we get $(w_l[k], \underline{w}_l[k]) = (A\cos\varphi,\ A\sin\varphi\cos\theta,\ A\sin\varphi\sin\theta)$.

**Local Orientation**: $\phi_l(k) = \arg(\underline{w}_l(k))$

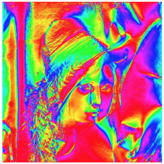
**Local Amplitude**: $A_l(k) = |(w_l[k], \underline{w}_l[k])|$

**Local Phase**: $\phi_l(k) = \arctan\left(\frac{|\underline{w}_l(k)|}{w_l(k)}\right)$

(a) Monogenic local amplitude of Lena

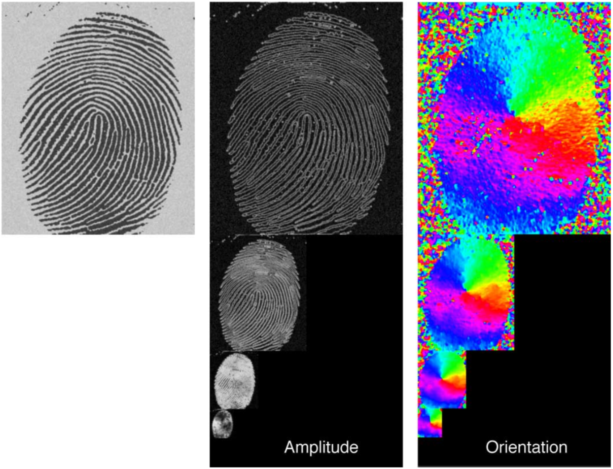


(b) Monogenic local phase



(c) Monogenic local orientation

# WAVELET ANALYSIS



# HIGHER ORDER RIESZ TRANSFORMS

$$\mathcal{R}_{i_1}\mathcal{R}_{i_2}\cdots\mathcal{R}_{i_N}f, \quad i_1, i_2, \cdots, i_N \in \{1, \cdots, n\}.$$

## Theorem

*The N-th Riesz transform achieves the following decomposition of the identits*

$$\sum_{|\underline{m}|=N} \frac{N!}{\underline{m}!}(\mathcal{R}_1^{m_1}\mathcal{R}_2^{m_2}\cdots\mathcal{R}_n^{m_n})^*(\mathcal{R}_1^{m_1}\mathcal{R}_2^{m_2}\cdots\mathcal{R}_n^{m_n}) = Id$$

*using the multi-index vactor* $\underline{m} = (m_1, \ldots, m_n)$.

The $N$-th-order Riesz tranform decomposes the signal into $\begin{pmatrix} N+n-1 \\ n-1 \end{pmatrix}$ distinct components and preseves energy, if

$$\mathcal{R}^{\underline{m}}f := \sqrt{\frac{N!}{m_1!\, m_2! \ldots m_n!}}\mathcal{R}_1^{m_1}\cdots\mathcal{R}_n^{m_n}f$$

then $\sum_{|\underline{m}|=N}\langle\mathcal{R}^{\underline{m}}f, \mathcal{R}^{\underline{m}}g\rangle_{L^2} = \langle f, g\rangle_{L^2}$.

# TEAGER-KAISER-ENERGY OPERATOR – TKEO

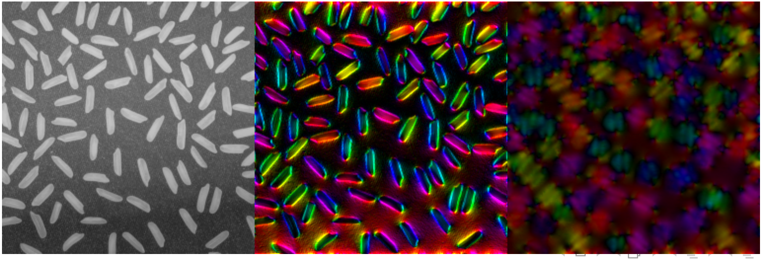
TKEO and energy $E$ for an $L^2(\mathbb{R}^2)$ signal:

$$E\{f\} = \begin{vmatrix} \mathcal{D}f & \mathcal{D}(\mathcal{D}f) \\ f & \mathcal{D}f \end{vmatrix}, \quad \mathcal{D} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y},$$

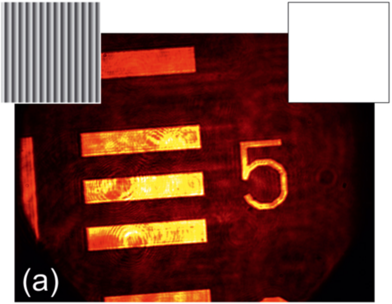
$$E\{f\} = [(f_x)^2 - (f_y)^2 - f(f_{xx} - f_{yy})] + 2i[f_x f_y - f\, f_{xy}].$$

Riesz transform based energy operator: $\mathcal{D}^{\mathcal{R}} = \mathcal{R}^x + i\mathcal{R}^y$

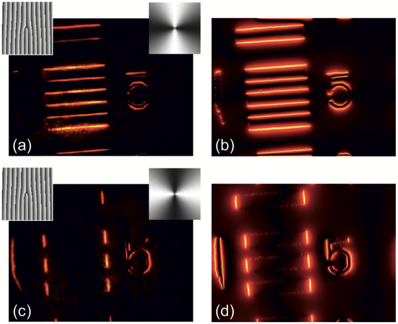
$$E^{\mathcal{R}}\{f\} = [(\mathcal{R}^x)^2 - (\mathcal{R}^y)^2 - f(\mathcal{R}^{xx} - \mathcal{R}^{yy})] + 2i[\mathcal{R}^x\mathcal{R}^y - f\,\mathcal{R}xy].$$



# STEERABILITY




(a) original




(a), (c) measured,
(b), (d) Riesz transforms

# MULTIORDER GENERALIZED RIESZ TRANSFORMS

Let $U_{M,N}$ be a matrix of size $M \times (2N+1)$ with $M \geq 1$.

## Definition

*The multiorder generalized Riesz transform with coefficient matrix $U_{M,N}$ is the scalar to M-vactor signal transform $\mathcal{R}_{M,N}f$ whose mth component is given by*

$$[\mathcal{R}_{U_{M,N}} f(\underline{x})]_m = \sum_{n=-N}^{N} u_{m,n} \mathcal{R}^n f(\underline{x}).$$

*Where*

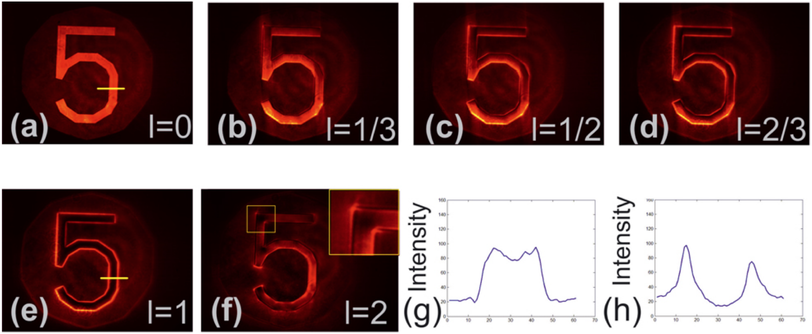
$$\mathcal{R}^n f(x,y) = (-\Delta)^{-n/2} \sum_{n_1=0}^{n} \binom{n}{n_1} (-i)^{n_1} \partial_x^{n_1} \partial_y^{n-n_1} f(x,y).$$

Properties:

- $\partial_{\underline{x}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$, $\partial_{\underline{x}} f(\underline{x}) = i\mathcal{R}(-\Delta)^{1/2} f(\underline{x})$ and $\mathcal{R}f(\underline{x}) = -i(-\Delta)^{-1/2}\partial_{\underline{x}} f(\underline{x})$.
- norm preservation, self-invertibility
- steerability: The generalized multiorder Riesz transform is steerable in the sense that its component impulse responses can be simultaneously rotated to any spatial orientation by forming suitable linear combinations.

# Fractional Riesz-Hilbert transform

- The fractional Hilbert transform by Lohmann is widly used in optics and image processing.
- The monogenic signal is the higher dimensional version of an analytic signal.
- Example $n = 2$ : $\left(\frac{\omega}{|\omega|}\right)^{\alpha} \sim e^{i\frac{\pi}{2}\alpha}$ (spiral filter by Larkin).



# How to define a fractional operator?

What we would like to have: $\mathcal{T}$ is a transform defined in an appropriate function space, usually a Hilbert space.

- $\mathcal{T}^1 = \mathcal{T}$,
- $\mathcal{T}^{-1} = \mathcal{T}^{-1}$ inverse transform,
- $\mathcal{T}^0 = I$ identiy operator,
- $\mathcal{T}^\alpha \mathcal{T}^\beta = \mathcal{T}^{\alpha+\beta}$, semigroup property.
- $\mathcal{T} f(\xi) = \int K_\alpha(\xi, x) f(x)\, dx$, closed form.
- How can we get that? Spectral analysis.

# SPECTRAL ANALYSIS

- $\mathcal{T} : H \to H$, where $H$ is a complex separable Hilbert with inner product $\langle . , . \rangle$,
- and if there is a complete set of orthonormal eigenfunctions $\phi_n$ with corresponding eigenvalues $\lambda_n$,
- then any element in the space can be represented as

$$f = \sum_{n=0}^{\infty} a_n \phi_n, \quad a_n = \langle f, \phi_n \rangle, \quad \text{so that} \quad \mathcal{T} f = \sum_{n=0}^{\infty} a_n \lambda_n \phi_n.$$

- The fractional transform can be defined as

$$(\mathcal{T}^a f)(\xi) = \sum_{n=0}^{\infty} a_n \lambda_n^a \phi_n(\xi) = \sum_{n=0}^{\infty} \lambda_n^a \langle f, \phi_n \rangle \phi_n(\xi) = \langle f, K_a(\xi, .) \rangle,$$

where

$$K_a(\xi, x) = \sum_{n=0}^{\infty} \overline{\lambda_n^a}\, \overline{\phi_n(\xi)}\, \phi_n(x).$$

- Problem: The Hilbert transform is not compact!

# First construction

We have

$$\varphi = \frac{1}{2}(I + H)\varphi + \frac{1}{2}(I - H)\varphi$$

and

$$H\varphi = \frac{1}{2}(I + H)\varphi + (-1)\frac{1}{2}(I - H)\varphi.$$

Therefore we define

$$\begin{aligned} H^{\alpha}\varphi &= \frac{1}{2}(I + \mathcal{H})\varphi + e^{-i\pi\alpha}\frac{1}{2}(I - H)\varphi \\ &= e^{-i\frac{\pi}{2}\alpha}\frac{1}{2}\left(e^{i\frac{\pi}{2}\alpha}(I + H)\varphi + e^{-i\frac{\pi}{2}\alpha}(I - H)\right)\varphi \\ &= e^{-i\frac{\pi}{2}\alpha}\left(\cos\left(\frac{\pi}{2}\alpha\right) I + i\sin\left(\frac{\pi}{2}\alpha\right) H\right)\varphi \end{aligned}$$

# FRACTIONAL RIESZ-HILBERT TRANSFORM

## Definition (First definition)

*The **fractional Riesz-Hilbert transform** is defined as*

$$H^{\alpha} = e^{-i\frac{\pi}{2}\alpha}\left(\cos\left(\frac{\pi}{2}\alpha\right) I + i\sin\left(\frac{\pi}{2}\alpha\right) H\right)$$

$$\mathcal{H}^{\alpha} = (iH)^{\alpha} = \cos\left(\frac{\pi}{2}\alpha\right) I + i\sin\left(\frac{\pi}{2}\alpha\right) H$$

Therefore the fractional Riesz-Hilbert transform is a linear combination of the identity operator and the Riesz-Hilbert operator.

# $C_0$-SEMIGROUP

## Definition (Semigroup)

*A family $T = \{T_t\}_{t\geq 0}$ of bounded linear operators acting on a Banach space $E$ is called a $C_0$-semigroup if the following three properties are satisfied:*

1. $T_0 = I$,
2. $T_{s+t} = T_s T_t$ *for all* $s,\ t \geq 0$,
3. $\lim_{t\to 0} T_t x = x$ *for all* $x \in E$.

*If the stronger condition* $\lim_{t\to 0} ||T_t - T||_E = 0$ *is satisfied the group is called norm continuous.*

# $C_0$-SEMIGROUP

We have

$$\begin{aligned} e^{i\frac{\pi}{2}\alpha H} &= \sum_{l=0}^{\infty} i^{2l}\left(\frac{\pi}{2}\alpha\right)^{2l}\frac{1}{(2l)!}H^{2l} + i^{2l+1}\left(\frac{\pi}{2}\alpha\right)^{2l+1}\frac{1}{(2l+1)!}H^{2l+1} \\ &= \sum_{l=0}^{\infty}(-1)^{l}\left(\frac{\pi}{2}\alpha\right)^{2l}\frac{1}{(2l)!}I + i(-1)^{l}\left(\frac{\pi}{2}\alpha\right)^{2l+1}\frac{1}{(2l+1)!}H \\ &= \cos\left(\frac{\pi}{2}\alpha\right)I + i\sin\left(\frac{\pi}{2}\alpha\right)H. \end{aligned}$$

and

$$\begin{aligned} e^{-i\pi\alpha\frac{1}{2}(I-H)} &= e^{-i\alpha\frac{\pi}{2}I}e^{i\alpha\frac{\pi}{2}H} \\ &= e^{-i\frac{\pi}{2}\alpha}\left(\cos(\frac{\pi}{2}\alpha) + i\sin(\frac{\pi}{2}\alpha)H\right). \end{aligned}$$

# FRACTIONAL RIESZ-HILBERT TRANSFORM

## Definition (Second definition)

$$H^{\alpha} = e^{-i\pi\alpha\frac{1}{2}(I-H)}, \quad \mathcal{H}^{\alpha} = e^{i\frac{\pi}{2}\alpha H}.$$

## Theorem

*The fractional Riesz-Hilbert transforms* $H^{\alpha}, \mathcal{H}^{\alpha} : L^p(\mathbb{R}^3, \mathbb{H}) \to L^p(\mathbb{R}^3, \mathbb{H}),\ 1 < p < \infty,\ \alpha \in \mathbb{R}$, *are linear, shift and scale invariant and fulfill the following properties*

1. *the inverse of* $H^{\alpha}$ *is* $H^{-\alpha}$ *and the inverse of* $\mathcal{H}^{\alpha}$ *is* $\mathcal{H}^{-\alpha}, \alpha \in \mathbb{R}$,
2. $H^{\alpha}$ *is* 2*-periodic in* $\alpha$, *i.e.* $H^{\alpha+2} = H^{\alpha}$, *whereas* $\mathcal{H}^{\alpha}$ *is* 4*-periodic in* $\alpha$, *i.e.* $\mathcal{H}^{\alpha+4} = \mathcal{H}^{\alpha},\ \alpha \in \mathbb{R}$.
3. *If* $f, g \in L^2(\mathbb{R}^3, \mathbb{R})$ *such that* $\langle f, g \rangle = 0$ *then*

$$\langle H^{\alpha} f, H^{\alpha} g \rangle = \langle \mathcal{H}^{\alpha} f, \mathcal{H}^{\alpha} g \rangle = 0.$$

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# THE END!

Thank you for your attentation.