Error Estimates and Convergence Rates for Filtered Back Projection

Matthias Beckmann and Armin Iske

Department of Mathematics, University of Hamburg

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Outline

Basic Reconstruction Problem

Problem formulation:

Let $\Omega \subset \mathbb{R}^2$ be bounded. Reconstruct a bivariate function $f \equiv f(x,y)$ with support $\mathrm{supp}(f) \subseteq \Omega$ from given Radon data

$$\big\{\mathcal{R}f(t,\theta) \mid t \in \mathbb{R},\, \theta \in [0,\pi)\big\},$$

where the **Radon transform** $\mathcal{R}f$ of $f \in \mathrm{L}^1(\mathbb{R}^2)$ is defined as

$$\mathcal{R}f(t,\theta) = \int_{\{x\cos(\theta)+y\sin(\theta)=t\}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{for } (t,\theta) \in \mathbb{R} \times [0,\pi).$$

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Analytical solution:

The inversion of $\mathcal{R}$ involves the **back projection** $\mathcal{B}h$ of $h \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$,

$$\mathcal{B}h(x,y) = \frac{1}{\pi}\int_0^\pi h(x\cos(\theta) + y\sin(\theta), \theta)\,\mathrm{d}\theta \qquad \text{for } (x,y) \in \mathbb{R}^2,$$

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and is given, for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{C}(\mathbb{R}^2)$, by the **filtered back projection formula**

$$f(x,y) = \frac{1}{2}\mathcal{B}\left(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S,\theta)]\right)(x,y) \quad \forall\,(x,y) \in \mathbb{R}^2.$$

Approximate Reconstruction

Stabilization: Replace the factor $|S|$ by a **low-pass filter** $A_L : \mathbb{R} \longrightarrow \mathbb{R}$,

$$A_L(S) = |S| W(S/L) = |S| W_L(S)$$

with finite bandwidth $L > 0$ and an *even* **window function** $W : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support $\mathrm{supp}(W) \subseteq [-1, 1]$.

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where we rely, for $f \in \mathrm{L}^1(\mathbb{R}^2)$ and $g \in \mathrm{L}^1(\mathbb{R} \times [0, \pi))$, on the standard relation

$$\mathcal{B} g * f = \mathcal{B}(g * \mathcal{R} f)$$

and define the **convolution kernel** $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as

$$K_L(x, y) = \frac{1}{2} \mathcal{B} \left(\mathcal{F}^{-1} A_L \right)(x, y) \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Analysis of the Reconstruction Error

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Definition (Sobolev space of fractional order)

The **Sobolev space** $\mathrm{H}^\alpha(\mathbb{R}^2)$ **of fractional order** $\alpha \in \mathbb{R}$ *is defined as*

$$\mathrm{H}^\alpha(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) \mid \|f\|_\alpha < \infty \right\},$$

where

$$\|f\|_\alpha^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(1 + x^2 + y^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

L^2-Error Analysis

Theorem (L^2-error estimate; see [Beckmann & Iske, 2015])

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$, $W \in \mathrm{L}^\infty(\mathbb{R})$ and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \|1 - W\|_{\infty,[-1,1]} \|f\|_{\mathrm{L}^2(\mathbb{R}^2)} + L^{-\alpha} \|f\|_\alpha.$$

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Theorem (Convergence in the L^p-norm, see [Madych, 1990])

Let the convolution kernel $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ with

$$\int_\mathbb{R} \int_\mathbb{R} K_L(x, y) \,\mathrm{d}x \,\mathrm{d}y = 1.$$

Then, for $f \in \mathrm{L}^p(\mathbb{R}^2)$, $1 \leq p < \infty$,

$$\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty.$$

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Refined Error Estimate

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ with $\alpha > 0$, $W \in \mathrm{L}^\infty(\mathbb{R})$ and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. By defining

$$W_L(x, y) = W_L(r(x, y))$$

for $r(x, y) = \sqrt{x^2 + y^2}$ and $(x, y) \in \mathbb{R}^2$, we have

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The integral I_2 is bounded above by

$$I_2 \leq \frac{1}{2\pi} \int_{r(x,y) > L} \left(1 + x^2 + y^2\right)^\alpha L^{-2\alpha} \, |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y) \leq L^{-2\alpha} \, \|f\|_\alpha^2.$$

Refined Error Estimate

The integral I_1 can be written as

$$I_1 = \frac{1}{2\pi} \int_{r(x,y) \leq L} \frac{\left|1 - W_L(x,y)\right|^2}{\left(1 + x^2 + y^2\right)^\alpha} \left(1 + x^2 + y^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y)$$

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and estimated by

$$I_1 \leq \left(\sup_{S \in [-L,L]} \frac{\left(1 - W_L(S)\right)^2}{\left(1 + S^2\right)^\alpha} \right) \frac{1}{2\pi} \int_{r(x,y) \leq L} \left(1 + x^2 + y^2\right)^\alpha \left|\mathcal{F}f(x,y)\right|^2 \mathrm{d}(x,y)$$

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For the sake of brevity, we define the function $\Phi_{\alpha,W,L} : [-1,1] \longrightarrow \mathbb{R}$ via

$$\Phi_{\alpha,W,L}(S) = \frac{\left(1 - W(S)\right)^2}{\left(1 + L^2 S^2\right)^\alpha} \quad \text{for } S \in [-1,1]$$

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and obtain

$$I_1 \le \left(\sup_{S \in [-1,1]} \Phi_{\alpha,W,L}(S) \right) \|f\|_\alpha^2 = \Phi_{\alpha,W}(L) \, \|f\|_\alpha^2.$$

Refined Error Estimate

Theorem (Refined L^2-error estimate)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$*, let* $W \in \mathrm{L}^\infty(\mathbb{R})$ *and* $K_L \in \mathrm{L}^1(\mathbb{R}^2)$*. Then, the* L^2*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\Phi_{\alpha,W}^{1/2}(L) + L^{-\alpha}\right)\|f\|_\alpha,$$

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Theorem (Convergence of $\Phi_{\alpha,W}$)

Let the window function W *be continuous on* $[-1,1]$ *and satisfy* $W(0) = 1$. *Then, for all* $\alpha > 0$,

$$\Phi_{\alpha,W}(L) = \max_{S\in[-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\alpha} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty. \qquad \square$$

Convergence of the Reconstruction Error

Corollary (L^2-Convergence of the reconstruction error)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ *and* $W \in \mathcal{C}([-1,1])$ *with* $W(0) = 1$*. Then, the* L^2*-norm of the FBP reconstruction error* $e_L = f - f_L$ *satisfies*

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Let $S^*_{\alpha,W,L} \in [0,1]$ denote the *smallest* maximizer of $\Phi_{\alpha,W,L}$ on $[0,1]$, i.e.,

$$\Phi_{\alpha,W}(L) = \sup_{S \in [-1,1]} \Phi_{\alpha,W,L}(S) = \Phi_{\alpha,W,L}(S^*_{\alpha,W,L}).$$

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Assumption

$S^*_{\alpha,W,L}$ is uniformly bounded away from 0, i.e., there exists a constant $c_{\alpha,W} > 0$ such that

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Under the above assumption follows that

$$\Phi_{\alpha,W}(L) \leq c_{\alpha,W}^{-2\alpha} \, \|1 - W\|^2_{\infty,[-1,1]} \, L^{-2\alpha} = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

Order of Convergence

Corollary (Rate of convergence)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$*, let* $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ *and* $W \in \mathcal{C}([-1,1])$ *with* $W(0) = 1$*. Further, let the above assumption be satisfied. Then, the* L^2*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c_{\alpha,W}^{-\alpha} \|1 - W\|_{\infty,[-1,1]} + 1\right) L^{-\alpha} \|f\|_\alpha.$$

Therefore,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}(L^{-\alpha}) \quad \text{for} \quad L \longrightarrow \infty,$$

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Example:

Let the window function $\chi_{[-1,1]} \neq W \in \mathcal{C}([-1,1])$ satisfy

$$W(S) = 1 \quad \forall \, S \in (-\varepsilon, \varepsilon)$$

with a constant $\varepsilon > 0$. Then, the above assumption is fulfilled with $c_{\alpha,W} = \varepsilon$.

Numerical Observations

We investigate the behaviour of $\Phi_{\alpha,W}$ numerically for the generalized Ramp filter $A_L(S) = |S|\, W(S/L)$ with the window function

$$W(S) = \begin{cases} 1 & , \; 0 \leq |S| \leq \beta \\ \frac{1}{1-\beta}\left(1 - \beta\gamma - (1-\gamma)\,|S|\right) & , \; \beta < |S| \leq 1 \end{cases} \qquad \text{for } S \in [-1, 1]$$

with width $\beta \in (0,1)$ and jump height $\gamma \in [0,1]$.

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Then, the above assumption

$$\exists\, c_{\alpha,W} > 0 \;\forall\, L > 0 : \; S^*_{\alpha,W,L} \geq c_{\alpha,W}$$

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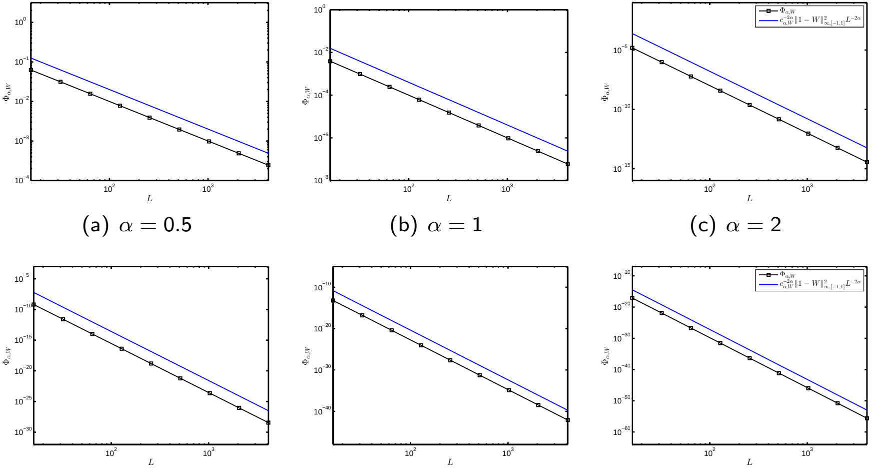
is fulfilled with the constant

$$c_{\alpha,W} = \beta.$$

Further, for all $\alpha > 0$, we observe that the convergence rate of $\Phi_{\alpha,W}$ is given by

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

Numerical Observations



(a) $\alpha = 0.5$ (b) $\alpha = 1$ (c) $\alpha = 2$

(d) $\alpha = 4$ (e) $\alpha = 6$ (f) $\alpha = 8$

Fig.: Decay rate of $\Phi_{\alpha,W}$ for the Ramp filter with width $\beta = 0.5$ and jump height $\gamma = 0$

Numerical Observations

We investigate the behaviour of $S^*_{\alpha,W,L}$ and $\Phi_{\alpha,W}$ numerically for the following low-pass filters:

- Shepp-Logan filter: $W(S) = \operatorname{sinc}\left(\frac{\pi S}{2}\right) \cdot \chi_{[-1,1]}(S)$,
- Cosine filter: $W(S) = \cos\left(\frac{\pi S}{2}\right) \cdot \chi_{[-1,1]}(S)$,
- Hamming filter (for $\beta \in \left[\frac{1}{2}, 1\right]$): $W(S) = (\beta + (1-\beta)\cos(\pi S)) \cdot \chi_{[-1,1]}(S)$,
- Gaussian filter (for $\beta > 1$): $W(S) = \exp\left(-(^{\pi S}/_{\beta})^2\right) \cdot \chi_{[-1,1]}(S)$.

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For $\alpha < 2$, we observe that the above assumption

$$\exists\, c_{\alpha,W} > 0\ \forall\, L > 0 :\ S^*_{\alpha,W,L} \geq c_{\alpha,W}$$

is fulfilled and

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is fulfilled and

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\alpha \geq 2$, we have

$$S^*_{\alpha,W,L} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty$$

and the convergence rate of $\Phi_{\alpha,W}$ stagnates at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty.$$

Numerical Observations

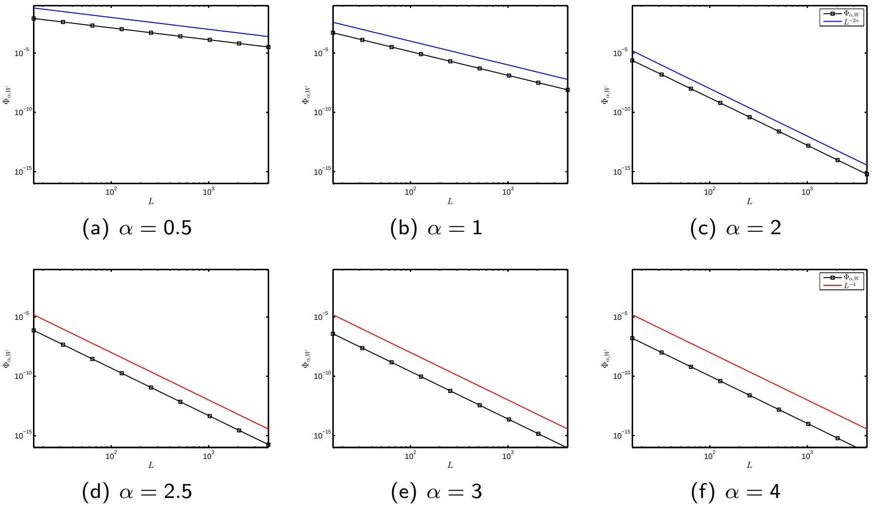

Fig.: Decay rate of $\Phi_{\alpha,W}$ for the Shepp-Logan filter

Error Analysis for $\mathcal{C}^2$-Windows

Theorem (Convergence rate of $\Phi_{\alpha,W}$ for $\mathcal{C}^2$-windows)

Let the window function W be twice continuously differentiable on $[-1,1]$ with $W(0)=1$ and let $\alpha>0$ be given. Then, we have

$$\Phi_{\alpha,W}(L) \leq \begin{cases} \frac{(\alpha-2)^{\alpha-2}}{\alpha^\alpha} \, \|W''\|^2_{\infty,[-1,1]} \, L^{-4} & , \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \frac{1}{4} \, \|W''\|^2_{\infty,[-1,1]} \, L^{-2\alpha} & , \alpha \leq 2 \vee \left(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\right) \end{cases}$$

for all $L>0$ and, especially,

$$\Phi_{\alpha,W}(L) = \mathcal{O}\left(L^{-\min\{4,2\alpha\}}\right) \quad \text{for} \quad L \longrightarrow \infty. \qquad \square$$

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Proof:

Let $S \in [-1,1]$ be fixed. Because W satisfies $W \in \mathcal{C}^2([-1,1])$ with $W(0)=1$, we can apply Taylor's theorem and obtain

$$W(S) = 1 + \frac{1}{2} W''(\xi) S^2$$

for some ξ between 0 and S, where we use $W'(0)=0$, since W is even.

Error Analysis for $\mathcal{C}^2$-Windows

This leads to

$$\Phi_{\alpha,W,L}(S) = \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \leq \frac{\|W''\|^2_{\infty,[-1,1]}}{4} \, \frac{S^4}{(1+L^2\,S^2)^\alpha}.$$

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Hence,

$$\Phi_{\alpha,W}(L) = \max_{S \in [-1,1]} \Phi_{\alpha,W,L}(S) \leq \frac{\|W''\|^2_{\infty,[-1,1]}}{4} \max_{S \in [-1,1]} \phi_{\alpha,L}(S)$$

and we now need to investigate

$$\phi_{\alpha,L}(S) = \frac{S^4}{\left(1 + L^2\, S^2\right)^\alpha} \quad \text{for } S \in [-1,1].$$

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We can show that

$$\max_{S\in[-1,1]} \phi_{\alpha,L}(S) = \begin{cases} \phi_{\alpha,L}(1) & , \alpha \leq 2 \vee \left(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\right) \\ \phi_{\alpha,L}\left(\frac{\sqrt{2}}{L\sqrt{\alpha-2}}\right) & , \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \end{cases}$$

$$\leq \begin{cases} L^{-2\alpha} & , \alpha \leq 2 \vee \left(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\right) \\ 4\,\frac{(\alpha-2)^{\alpha-2}}{\alpha^\alpha}\,L^{-4} & , \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}}. \end{cases}$$

□

Error Analysis for $\mathcal{C}^2$-Windows

Corollary (L^2-error estimate for $\mathcal{C}^2$-windows)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$, let $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ and $W \in \mathcal{C}^2([-1,1])$ with $W(0) = 1$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ satisfies

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \begin{cases} \left(\frac{c^2_{\alpha,2}}{4} \|W''\|^2_{\infty,[-1,1]} L^{-4} + L^{-2\alpha}\right) \|f\|^2_\alpha & , \alpha > 2 \wedge L \geq L^* \\ \left(\frac{1}{4} \|W''\|^2_{\infty,[-1,1]} L^{-2\alpha} + L^{-2\alpha}\right) \|f\|^2_\alpha & , \alpha \leq 2 \vee (\alpha > 2 \wedge L < L^*) \end{cases}$$

with the critical bandwidth $L^* = \frac{\sqrt{2}}{\sqrt{\alpha - 2}}$ for $\alpha > 2$ and the constant

$$c_{\alpha,2} = \frac{2}{\alpha - 2} \left(\frac{\alpha - 2}{\alpha}\right)^{\alpha/2},$$

which is strictly monotonically decreasing in $\alpha > 2$. Especially, we have

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c \, \|W''\|_{\infty,[-1,1]} L^{-\min\{2,\alpha\}} + L^{-\alpha}\right) \|f\|_\alpha = \mathcal{O}\left(L^{-\min\{2,\alpha\}}\right).$$ □

Error Analysis for $\mathcal{C}^k$-Windows

Theorem (Convergence rate of $\Phi_{\alpha,W}$ for $\mathcal{C}^k$-windows)

Let the window function W be k-times continuously differentiable on $[-1,1]$, $k \geq 2$, with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1$$

and let $\alpha > 0$ be given. Then, we have

$$\Phi_{\alpha,W}(L) \leq \begin{cases} \frac{c_{\alpha,k}^2}{(k!)^2} \, \|W^{(k)}\|_{\infty,[-1,1]}^2 \, L^{-2k} & , \alpha > k \wedge L \geq L^* \\ \frac{1}{(k!)^2} \, \|W^{(k)}\|_{\infty,[-1,1]}^2 \, L^{-2\alpha} & , \alpha \leq k \vee (\alpha > k \wedge L < L^*) \end{cases}$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha-k}}$ for $\alpha > k$ and the constant

$$c_{\alpha,k} = \left(\frac{k}{\alpha-k}\right)^{k/2} \left(\frac{\alpha-k}{\alpha}\right)^{\alpha/2}.$$

Especially,

$$\Phi_{\alpha,W}(L) = \mathcal{O}\left(L^{-2\min\{k,\alpha\}}\right) \quad \text{for} \quad L \longrightarrow \infty.$$

□

Error Analysis for $\mathcal{C}^k$-Windows

Corollary (L^2-error estimate for $\mathcal{C}^k$-windows)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$, let $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ and $W \in \mathcal{C}^k([-1,1])$, $k \geq 2$, with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ satisfies

$$\|e_L\|_{\mathrm{L}^2}^2 \leq \begin{cases} \left(\frac{c_{\alpha,k}^2}{(k!)^2} \|W^{(k)}\|_{\infty,[-1,1]}^2 L^{-2k} + L^{-2\alpha}\right) \|f\|_\alpha^2 & , \alpha > k \wedge L \geq L^* \\ \left(\frac{1}{(k!)^2} \|W^{(k)}\|_{\infty,[-1,1]}^2 L^{-2\alpha} + L^{-2\alpha}\right) \|f\|_\alpha^2 & , \alpha \leq k \vee (\alpha > k \wedge L < L^*) \end{cases}$$

and the constant

$$c_{\alpha,k} = \left(\frac{k}{\alpha - k}\right)^{k/2} \left(\frac{\alpha - k}{\alpha}\right)^{\alpha/2}$$

is strictly monotonically decreasing in $\alpha > k$. Especially, we have

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c\, \|W^{(k)}\|_{\infty,[-1,1]} L^{-\min\{k,\alpha\}} + L^{-\alpha}\right) \|f\|_\alpha = \mathcal{O}\left(L^{-\min\{k,\alpha\}}\right). \qquad \square$$

Numerical Results

We investigate the behaviour of $\Phi_{\alpha,W}$ numerically for the generalized Gaussian filter $A_L(S) = |S|\, W(S/L)$ with the window function

$$W(S) = \exp\left(- \left(\frac{\pi S}{\beta}\right)^k \right) \quad \text{for } S \in [-1, 1]$$

for $k \in \mathbb{N}_{\geq 2}$ and $\beta > 1$.

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for $k \in \mathbb{N}_{\geq 2}$ and $\beta > 1$. Then, W satisfies $W \in \mathcal{C}^k([-1,1])$ with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1, \qquad W^{(k)}(0) = -k!\left(\frac{\pi}{\beta}\right)^k \neq 0.$$

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For $\alpha < k$, we observe

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

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$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall \, 1 \leq j \leq k-1, \qquad W^{(k)}(0) = -k! \left(\frac{\pi}{\beta}\right)^k \neq 0.$$

For $\alpha < k$, we observe

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\alpha \geq k$, the convergence rate of $\Phi_{\alpha,W}$ stagnates at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2k}) \quad \text{for} \quad L \longrightarrow \infty.$$

Numerical Results

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for $k \in \mathbb{N}_{\geq 2}$ and $\beta > 1$. Then, W satisfies $W \in \mathcal{C}^k([-1,1])$ with

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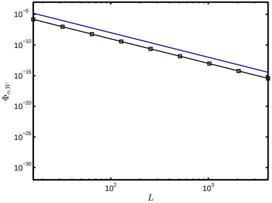
$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\alpha \geq k$, the convergence rate of $\Phi_{\alpha,W}$ stagnates at

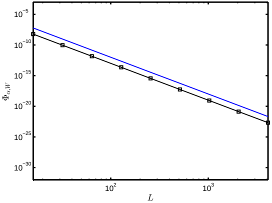
$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2k}) \quad \text{for} \quad L \longrightarrow \infty.$$

This shows that our proven convergence order of $\Phi_{\alpha,W}$ is optimal for $\mathcal{C}^k$-windows.

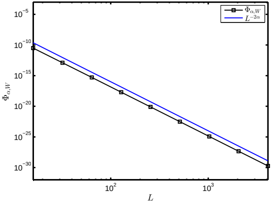
Numerical Results



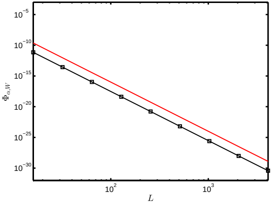
(a) $\alpha = 2$



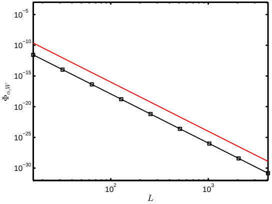
(b) $\alpha = 3$



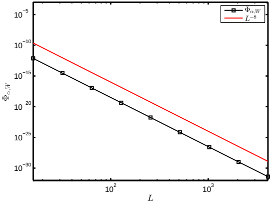
(c) $\alpha = 4$



(d) $\alpha = 4.5$



(e) $\alpha = 5$



(f) $\alpha = 6$

Fig.: Decay rate of $\Phi_{\alpha,W}$ for the generalized Gaussian filter with $k = 4$ and $\beta = 4$

Asymptotic L^2-Error Estimate

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ and let $W \in \mathrm{L}^\infty(\mathbb{R})$ be k-times differentiable at the origin, $k \geq 2$, with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k - 1.$$

Asymptotic L^2-Error Estimate

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$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

We again start with

$$\|e_L\|_{\mathrm{L}^2}^2 = \underbrace{\frac{1}{2\pi} \int_{r \leq L} |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \,\mathrm{d}(x,y)}_{=: I_1} + \underbrace{\frac{1}{2\pi} \int_{r > L} |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y)}_{=: I_2}.$$

Asymptotic L^2-Error Estimate

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ and let $W \in \mathrm{L}^\infty(\mathbb{R})$ be k-times differentiable at the origin, $k \geq 2$, with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

We again start with

$$\|e_L\|_{\mathrm{L}^2}^2 = \underbrace{\frac{1}{2\pi} \int_{r \leq L} |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \,\mathrm{d}(x,y)}_{=:I_1} + \underbrace{\frac{1}{2\pi} \int_{r > L} |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y)}_{=:I_2}.$$

As before, the integral I_2 can be bounded above by

$$I_2 = \frac{1}{2\pi} \int_{r(x,y) > L} |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y) \leq L^{-2\alpha} \|f\|_\alpha^2.$$

Asymptotic L^2-Error Estimate

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ and let $W \in \mathrm{L}^\infty(\mathbb{R})$ be k-times differentiable at the origin, $k \geq 2$, with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

We again start with

$$\|e_L\|_{\mathrm{L}^2}^2 = \underbrace{\frac{1}{2\pi} \int_{r \leq L} |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \, \mathrm{d}(x,y)}_{=: I_1} + \underbrace{\frac{1}{2\pi} \int_{r > L} |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y)}_{=: I_2}.$$

As before, the integral I_2 can be bounded above by

$$I_2 = \frac{1}{2\pi} \int_{r(x,y) > L} |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \leq L^{-2\alpha} \|f\|_\alpha^2.$$

For the integral I_1, we have

$$I_1 = \frac{1}{2\pi} \int_{r(x,y) \leq L} \left| 1 - W\left(\frac{r(x,y)}{L} \right) \right|^2 |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).$$

Asymptotic L^2-Error Estimate

Using Taylor's theorem and Lebesgue's theorem on dominated convergence, we get

$$I_1 \leq 2\,\phi^*_{\alpha,L,k}\left(\frac{W^{(k)}(0)}{k!}\right)^2 \|f\|^2_\alpha + \phi^*_{\alpha,L,k}\, o(1) \quad \text{for} \quad L \longrightarrow \infty,$$

where

$$\phi^*_{\alpha,L,k} = \max_{S\in[0,1]} \frac{S^{2k}}{(1+L^2\,S^2)^\alpha} = \max_{S\in[0,1]} \phi_{\alpha,L,k}(S).$$

Asymptotic L^2-Error Estimate

Using Taylor's theorem and Lebesgue's theorem on dominated convergence, we get

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The maximum $\phi^*_{\alpha,L,k}$ of the function $\phi_{\alpha,L,k}$ on $[0,1]$ can be bounded by

$$\phi^*_{\alpha,L,k} \leq \begin{cases} c^2_{\alpha,k}\,L^{-2k} & ,\ \alpha > k \wedge L \geq L^* \\ L^{-2\alpha} & ,\ \alpha \leq k \vee (\alpha > k \wedge L < L^*) \end{cases} = \mathcal{O}\left(L^{-2\min\{k,\alpha\}}\right)$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha-k}}$ for $\alpha > k$ and the constant

$$c_{\alpha,k} = \left(\frac{k}{\alpha-k}\right)^{k/2}\left(\frac{\alpha-k}{\alpha}\right)^{\alpha/2}.$$

Asymptotic L^2-Error Estimate

Theorem (Asymptotic L^2-error estimate)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ *and* $W \in \mathrm{L}^\infty(\mathbb{R})$ *be* k*-times differentiable at the origin,* $k \geq 2$, *with*

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, the L^2*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \begin{cases} \left(\frac{\sqrt{2}}{k!}\, c_{\alpha,k} |W^{(k)}(0)|\, L^{-k} + L^{-\alpha}\right) \|f\|_\alpha + o(L^{-k}) & , \alpha > k \wedge L \geq L^* \\ \left(\frac{\sqrt{2}}{k!}\, |W^{(k)}(0)|\, L^{-\alpha} + L^{-\alpha}\right) \|f\|_\alpha + o(L^{-\alpha}) & , \alpha \leq k \vee (\alpha > k \wedge L < L^*) \end{cases}$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha - k}}$ *for* $\alpha > k$ *and the constant*

$$c_{\alpha,k} = \left(\frac{k}{\alpha - k}\right)^{k/2} \left(\frac{\alpha - k}{\alpha}\right)^{\alpha/2}.$$

Especially, we have

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c\, |W^{(k)}(0)|\, L^{-\min\{k,\alpha\}} + L^{-\alpha}\right) \|f\|_\alpha + o\left(L^{-\min\{k,\alpha\}}\right). \qquad \square$$

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Thank you for your attention!