Lower Estimates for Smooth Data Approximation

IM-Workshop on Signals, Images, and Approximation

Bernried – February 29, 2016

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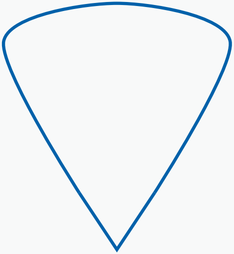
Motivation

We want to localize the singularities of a piecewise smooth function:



function f

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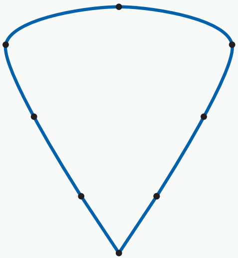

function	f
samples	$\alpha_1^*(f), \ldots, \alpha_n^*(f)$

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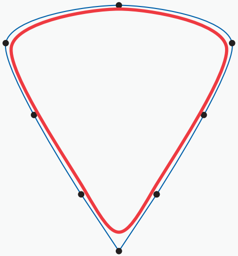

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smooth approximation	$Tf = \sum_{k=1}^{n} \alpha_k^*(f) e_k$

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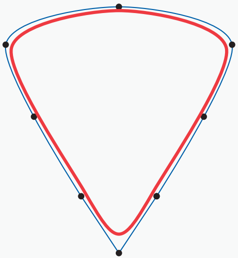

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samples	$\alpha_1^*(f), \ldots, \alpha_n^*(f)$
smooth approximation	$Tf = \sum_{k=1}^n \alpha_k^*(f) e_k$

Idea

relate the decay rate of the approximation error with the smoothness of f

Outline

Setting

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Banach function spaces

$\Omega \subset \mathbb{R}^d$ open set

$$X_p^0(\Omega) := L_p(\Omega), \quad 1 \le p < \infty; \qquad X_\infty^0(\Omega) := C(\Omega)$$

$$X_p^r(\Omega) := W_p^r(\Omega), \quad 1 \le p < \infty; \qquad X_\infty^r(\Omega) := C^r(\Omega)$$

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K-functional

$f \in X_p^0(\Omega)$

$$K_r^p(f, t^r) := \inf \left\{ \|f - g\|_p + t^r \, |g|_{r,p} \; : \; g \in X_p^r(\Omega) \right\}$$

with semi-norms

$$|g|_{r,p} := \sup_{|\alpha|=r} \|D^\alpha g\|_p \,, \qquad D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

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Sequence of smooth approximation operators

$\|T_n\|_{op} \le 1$

$$T_n : X_p^0(\Omega) \to X_p^r(\Omega) \qquad \text{with } \|T_n f - f\|_p \to 0.$$

Lower estimates

Question 1

Can we achieve a lower estimate, such that for all $f \in X_p^0(\Omega)$

$$\inf\left\{\|g-f\|_p + t_n^r\,|g|_{r,p} \;:\; g \in X_p^r(\Omega)\right\} \le C_n \cdot \|T_n f - f\|_p\,?$$

Lower estimates

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Question 2

Can we achieve a lower estimate, such that for all $f \in X_p^0(\Omega)$

$$\|T_n f - f\|_p + t_n^r \, |T_n f|_{r,p} \leq C_n \cdot \|T_n f - f\|_p?$$

Lower estimates

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Question 2

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Idea

estimate the semi-norms by the approximation error:

$$|T_n f|_{r,p} \le \tilde{C}_n \cdot \|T_n f - f\|_p$$

Estimate semi-norms by $\|Tf - f\|_p$

Denote by

$$\gamma := \sup\{|\lambda| \ : \ \lambda \in \sigma(T) \quad \text{with} \quad |\lambda| < 1\}.$$

Theorem [N., 2015]

Suppose

1. there exists P such that $\|T^m - P\|_{op} \leq C \cdot \gamma^m$,
2. $D^\alpha P = 0$ for all α with $|\alpha| = r$, and
3. D^α is bounded on $\operatorname{ran}(T)$ for all α with $|\alpha| = r$.

Then

$$|Tf|_{r,p} \leq \frac{\max_{|\alpha|=r} \|D^\alpha\|_{op:\operatorname{ran}(T)}}{1-\gamma} \|Tf - f\|_p .$$

Estimate semi-norms by $\|Tf - f\|_p$

1. $T^m \to P$
2. $D^\alpha P = 0$
3. $\|D^\alpha\|_{op:\mathrm{ran}(T)} < \infty$

We can estimate:

$$|Tf|_{r,p} = \sup_{|\alpha|=r} \|D^\alpha Tf\|_p$$

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$$|Tf|_{r,p} = \sup_{|\alpha|=r} \left\| D^\alpha T f - D^\alpha T^2 f + D^\alpha T^2 f - D^\alpha T^3 f + \cdots \right\|_p$$

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$$|Tf|_{r,p} = \sup_{|\alpha|=r} \left\| \sum_{m=1}^{\infty} D^\alpha T^m (f - Tf) \right\|_p$$

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&\leq \sup_{|\alpha|=r} \|D^\alpha\|_{op:\mathrm{ran}(T)} \cdot \|Tf - f\|_p \cdot \sum_{m=0}^{\infty} \gamma^m
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Conditions on T

1. $T^m \to P$
2. $D^\alpha P = 0$
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$$T^m \to P$$

Under which assumptions do the iterates converge?

The fundamental work of Dunford

Theorem [Dunford, 1943]

Let T be an operator such that $\|T\|_{op} \leq 1$ and

$$\left\|T^{m+1} - T^m\right\|_{op} \to 0, \qquad m \to \infty.$$

Then the following statements are equivalent:

1. $T^m \to P$ uniformly, $P^2 = P$ and $\operatorname{ran}(P) = \ker(T - \mathrm{Id})$.
2. $X = \ker(T - \mathrm{Id}) \oplus \operatorname{ran}(T - \mathrm{Id})$ and $\operatorname{ran}(T - \mathrm{Id})$ is closed.
3. The point 1 is either in $\rho(T)$ or else a simple pole of $(T - \mathrm{Id})^{-1}$.

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All items hold true if T^n is compact for some $n \in \mathbb{N}$.

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Open Questions

- When does $\left\|T^{m+1} - T^m\right\|_{op}$ converge?
- How to derive P?

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The theorem of Katznelson and Tzafriri

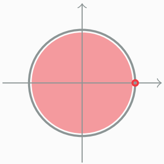
Theorem [Katznelson and Tzafriri, 1986]

Let T be an operator such that $\|T\|_{op} \leq 1$. Then

$$\lim_{m \to \infty} \left\| T^{m+1} - T^{m} \right\|_{op} = 0$$

if and only if

$$\sigma(T) \subset B(0,1) \cup \{1\}\,.$$



Y. Katznelson and L. Tzafriri. On power bounded operators. 1986.

Positive linear operators with finite rank

X Banach function space with $1 \in X$ and $\|1\| = 1$.

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Positive linear operators with finite rank

X Banach function space with $1 \in X$ and $\|1\| = 1$.

e.g., $C([0,1]^d)$, $L_p([0,1]^d)$

Given positive functions $e_1, \dots, e_n \in X$ such that

$$\sum_{k=1}^{n} e_k = 1,$$

we define the sequence of approximation operators by

$$T_n f := \sum_{k=1}^{n} \alpha_k^*(f)\, e_k, \qquad f \in X,$$

where α_k^* are normalized positive linear functionals satisfying $\alpha_k^*(e_k) > 0$.

Spectral location

Theorem [N., 2015]

Let T_n be the positive finite-rank operator defined previously, then

$$\sigma(T) \subset B(0,1) \cup \{1\}$$

and $P := \lim_{m\to\infty} T^m$ exists. P is the projection onto $\ker(T - \mathrm{Id})$.

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Idea of the proof

- all non-zero eigenvalues of T are eigenvalues of the Gramian matrix

$$\begin{pmatrix} \alpha_1^*(e_1) & \alpha_1^*(e_2) & \cdots & \alpha_1^*(e_n) \\ \alpha_2^*(e_1) & \alpha_2^*(e_2) & \cdots & \alpha_2^*(e_n) \\ \vdots & & & \\ \alpha_n^*(e_1) & \alpha_n^*(e_2) & \cdots & \alpha_n^*(e_n) \end{pmatrix}$$

- the matrix is nonnegative and its rows sum up to one
- the stated property follows using Gershgorin circles

S. A. Gershgorin. Über die Abgrenzung der Eigenwerte einer Matrix. 1931.

$$T^m \to P$$

How to derive P?

Related Work

- Kelisky and Rivlin (1967)
- Karlin and Ziegler (1970)
- C. A. Micchelli (1973)
- J. Nagel (1980)
- H. J. Wenz (1997)
- O. Agratini (2002)
- I. A. Rus (2004)
- C. Badea (2009)
- Gavrea and Ivan (2010, 2011, 2011)
- Altomare (2013)

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How to derive P?

We consider the fixed-point space of T

$$M = \ker(T - \mathrm{Id}) = \{x \in X \,:\, Tx = x\}$$

with normalized basis $\{e_1, \ldots, e_n\}$.

How to derive P?

We consider the fixed-point space of T

$$M = \ker(T - \mathrm{Id}) = \{x \in X \ : \ Tx = x\}$$

with normalized basis $\{e_1, \dots, e_n\}$. Then every $x \in M$ has a unique representation

$$x = \sum_{i=1}^{n} \alpha_i^*(x) e_i,$$

where a_i^* are appropriate continuous linear functionals on M.

We define the composition operator $\Phi : \mathbb{C}^n \to M$ and the decomposition operator $\Phi^* : M \to \mathbb{C}^n$ by

$$\Phi(a_1, \dots, a_n) = \sum_{i=1}^{n} a_i e_i, \qquad \Phi^*(x) = \begin{pmatrix} \alpha_1^*(x) \\ \vdots \\ \alpha_n^*(x) \end{pmatrix}.$$

The classical coordinate map

Then

$$\Phi^*\Phi = \begin{pmatrix} \alpha_1^*(e_1) & \cdots & \alpha_1^*(e_n) \\ \vdots & & \vdots \\ \alpha_n^*(e_1) & \cdots & \alpha_n^*(e_n) \end{pmatrix} = I_n \quad \text{and} \quad \Phi\Phi^* = \mathrm{Id}_M \,.$$

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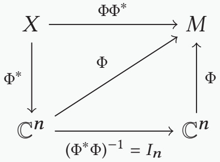

Proposition

The operator $\Phi\Phi^* : M \to M$ can be extended to a projection on X onto M and

$$X = \ker(T - \mathrm{Id}) \oplus \ker(\Phi\Phi^*).$$

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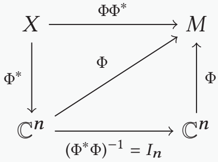

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$= \mathrm{ran}(T - \mathrm{Id})$?

How to choose the linear functionals?

Let $\Lambda \subset X^*$ such that $n = \dim(M) = \dim(\Lambda)$.
Is it possible to choose functionals in Φ^* only from the set Λ?

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Lemma

$$\begin{array}{ccc} X & \xrightarrow{\Phi(\Phi^*\Phi)^{-1}\Phi^*} & M \\ {\scriptstyle \Phi^*}\downarrow & & \uparrow{\scriptstyle \Phi} \\ \mathbb{C}^n & \xrightarrow[(\Phi^*\Phi)^{-1}]{} & \mathbb{C}^n \end{array}$$

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Lemma

Let $\{e_1^*, \ldots, e_n^*\}$ form a basis for Λ. Then the operator $P : X \to X$,

$$Px := \Phi A \Phi^*(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} e_j^*(x) e_i, \qquad x \in X,$$

yields a projection onto M if and only if the matrix

$$G := (\Phi^* \Phi) = \begin{pmatrix} e_1^*(e_1) & \cdots & e_1^*(e_n) \\ \vdots & & \vdots \\ e_n^*(e_1) & \cdots & e_n^*(e_n) \end{pmatrix} \in \mathbb{C}^{n \times n}$$

is invertible. In this case $A = G^{-1}$.

When does $\ker(P) = \operatorname{ran}(T - \mathrm{Id})$ hold?

We consider

$$M = \ker(T - \mathrm{Id}) = \{x \in X \ : \ Tx = x\} \subset X,$$
$$\Lambda = \ker(T^* - \mathrm{Id}) = \{x^* \in X^* \ : \ T^* x^* = x^*\} \subset X^*,$$

with normalized bases

$$M = \operatorname{span}\{e_1, \ldots, e_n\} \text{ and } \Lambda = \operatorname{span}\left\{e_1^*, \ldots, e_n^*\right\}.$$

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Lemma [N., 2015]

Then the Gram matrix $G := (\Phi^* \Phi)$ is invertible if and only if $T - \mathrm{Id}$ has ascent one, i.e.,

$$\ker(T - \mathrm{Id}) = \ker(T - \mathrm{Id})^2 = \cdots .$$

When does $\ker(P) = \operatorname{ran}(T - \mathrm{Id})$ hold?

Theorem [N., 2015]

Let T be such that $\dim\ker(T - \mathrm{Id}) = \dim\ker(T^* - \mathrm{Id}) < \infty$.

Then the following statements are equivalent:

1. $T - \mathrm{Id}$ has ascent one.
2. $\Phi^*\Phi$ is invertible.
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We have obtained the space decomposition

$$X = \ker(T - \mathrm{Id}) \oplus \operatorname{ran}(T - \mathrm{Id}) = \operatorname{ran}(P) \oplus \ker(P)$$

with the projection operator

$$Px = \Phi(\Phi^*\Phi)^{-1}\Phi^*(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} e_j^*(x) e_i, \qquad x \in X.$$

Conditions on T

1. $\dim(\ker(T - \mathrm{Id})) > 0$
2. $\dim(\ker(T - \mathrm{Id}) = \dim\ker(T^* - \mathrm{Id}) < \infty$
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Note

- The second items holds true if T is a compact operator.
- The first and last item are guaranteed for positive compact operators with $r(T) = \|T\|_{op}$.

M. G. Krein and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. 1948.
H. P. Lotz. Über das Spektrum positiver Operatoren. 1968.

Summary

We have

- shown a general framework to prove lower estimates,
- characterized three sufficient criteria by

1. $T^m \to P$ — iterates converge
2. $D^\alpha P = 0$ — $|\cdot|_{r,p}$ annihilates fixed points of T
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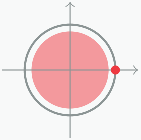
The convergence criteria of the iterates can be characterized by

- spectral location and
- the invertibility of a Gramian matrix.



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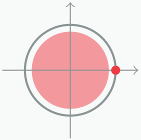
- characterized three sufficient criteria by
  1. $T^m \to P$ — iterates converge
  2. $D^\alpha P = 0$ — $|\cdot|_{r,p}$ annihilates fixed points of T
  3. $\|D^\alpha\|_{op:\mathrm{ran}(T)} < \infty$ — range of T ist smooth

The convergence criteria of the iterates can be characterized by

- spectral location and
- the invertibility of a Gramian matrix.



The iterates of positive linear operators of finite-rank always converge.

$$T^{\infty}$$

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