Wavelet frames and the unitary extension principle

An Invitation

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Frames

If a sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert spaces $\mathcal{H}$ is a frame, there exists another frame $\{g_k\}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \; f \in \mathcal{H}.$$

Similar to the decomposition in terms of an orthonormal basis, but MUCH MORE flexible.

Plan for the talk

- Frames and dual pairs of frames $\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty}$ in general Hilbert spaces $\mathcal{H}$, and the associated expansion

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \; f \in \mathcal{H}.$$

- Wavelet frames in $L^2(\mathbb{R})$
  - The unitary extension principle by Ron & Shen;
  - Applications and generalizations;
  - Complex pseudosplines and construction of wavelet frames (joint work with Brigitte Forster and Peter Massopust).

Frames

Definition: A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a *frame* if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\,||f||^2, \ \forall f \in \mathcal{H}.$$

A and B are called *frame bounds. The frame is tight if we can choose* $A = B$.

Note:

- Any orthonormal basis is a frame;
- Example of a frame which is not a basis:

$$\{e_1, e_1, e_2, e_3, \dots\},$$

where $\{e_k\}_{k=1}^{\infty}$ is an ONB. A frame can be redundant!

The frame decomposition

If $\{f_k\}_{k=1}^{\infty}$ is a frame, the frame operator

$$S : \mathcal{H} \to \mathcal{H},\ Sf = \sum \langle f, f_k \rangle f_k$$

is well-defined, bounded, invertible, and selfadjoint.

Theorem - the frame decomposition Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S. Then

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k, \ \ \forall f \in \mathcal{H}.$$

It might be difficult to compute S^{-1} !

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Important special case: If the frame $\{f_k\}_{k=1}^{\infty}$ is tight, $A = B$, then $S = A\,I$ and

$$f = \frac{1}{A} \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \;\; \forall f \in \mathcal{H}.$$

General dual frames

A frame which is not a basis is said to be *overcomplete*.

Theorem: *Assume that $\{f_k\}_{k=1}^{\infty}$ is an overcomplete frame. Then there exist frames*

$$\{g_k\}_{k=1}^{\infty} \neq \{S^{-1}f_k\}_{k=1}^{\infty}$$

for which

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, S^{-1}f_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

$\{g_k\}_{k=1}^{\infty}$ is called a *dual frame* of $\{f_k\}_{k=1}^{\infty}$. The special choice

$$\{g_k\}_{k=1}^{\infty} = \{S^{-1}f_k\}_{k=1}^{\infty}$$

is called the *canonical dual frame.*

Key tracks in frame theory:

- Frames in finite-dimensional spaces;
- Frames in general separable Hilbert spaces
- Concrete frames in concrete Hilbert spaces:
  - Gabor frames in $L^2(\mathbb{R}), L^2(\mathbb{R}^d)$;
  - Wavelet frames;
  - Shift-invariant systems, generalized shift-invariant (GSI) systems;
  - Shearlets, etc.
- Frames in Banach spaces;
- (GSI) Frames on LCA groups
- Frames via integrable group representations, coorbit theory.

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Research Group HATA DTU (Harmonic Analysis - Theory and Applications , Technical University of Denmark),

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An Introduction to frames and Riesz bases, 2.edition, Birkhäuser 2016

Classical wavelet theory

- Consider the translation operators T_k and scaling operators D, acting on functions $f \in L^2(\mathbb{R})$ by

$$T_k f(x) = f(x-k), k \in \mathbb{Z}, \qquad Df(x) = 2^{1/2} f(2x).$$

- Given a function $\psi \in L^2(\mathbb{R})$ and $j, k \in \mathbb{Z}$, consider

$$D^j T_k \psi(x) = 2^{j/2} \psi(2^j x - k),\ x \in \mathbb{R}.$$

- If $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, the function ψ is called a *wavelet*. In this case every $f \in L^2(\mathbb{R})$ has the representation

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, D^j T_k \psi \rangle D^j T_k \psi.$$

Multiresolution analysis - a tool to construct a wavelet

Definition: *A multiresolution analysis for* $L^2(\mathbb{R})$ *consists of a sequence of closed subspaces* $\{V_j\}_{j\in\mathbb{Z}}$ *of* $L^2(\mathbb{R})$ *and a function* $\phi \in V_0$*, such that the following conditions hold:*

(i) $\cdots V_{-1} \subset V_0 \subset V_1 \cdots$.

(ii) $\overline{\cup_j V_j} = L^2(\mathbb{R})$ *and* $\cap_j V_j = \{0\}$.

(iii) $f \in V_j \Leftrightarrow [x \to f(2x)] \in V_{j+1}$.

(iv) $f \in V_0 \Rightarrow T_k f \in V_0, \ \forall k \in \mathbb{Z}$.

(v) $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* V_0.

Construction of wavelet ONB

Theorem: Let $\phi \in L^2(\mathbb{R})$, and let $V_j := \overline{\text{span}}\{D^j T_k \phi\}_{k\in\mathbb{Z}}$. Assume that the following conditions hold:

(i) $\inf_{\gamma\in]-\epsilon,\epsilon[} |\hat{\phi}(\gamma)| > 0$ for some $\epsilon > 0$;

(ii) The scaling equation

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma),$$

is satisfied for a bounded 1-periodic function H_0;

(iii) $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system.

Then ϕ generates a multiresolution analysis, and there exists a wavelet ψ such that

$$\widehat{\psi}(2\gamma) = H_1(\gamma)\widehat{\phi}(\gamma)$$

with $H_1(\gamma) = \overline{H_0(\gamma + 1/2)}e^{-2\pi i\gamma}$. Explicitly, with $H_1(\gamma) = \sum_{k\in\mathbb{Z}} d_k e^{2\pi i\gamma}$,

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} d_k \phi(2x + k).$$

Spline wavelets B_N

- The B-splines $B_N, \; N \in \mathbb{N}$, are given by

$$B_1 = \chi_{[-1/2,1/2]}, \;\; B_{N+1} = B_N * B_1.$$

- One can consider even order splines B_N and define associated multiresolution analyses, which leads to wavelets of the type

$$\psi(x) = \sum_{k \in \mathbb{Z}} c_k B_N(2x+k).$$

- These wavelets are called *Battle–Lemarié wavelets.*
- Only shortcoming: all coefficients c_k are non-zero, which implies that the wavelet ψ has support equal to $\mathbb{R}$.

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Can show:

- There does not exists an ONB $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$ for $L^2(\mathbb{R})$ generated by a finite linear combination

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- There does not exists a tight frame $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$ for $L^2(\mathbb{R})$ generated by a finite linear combination

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- There does not exists a pairs of dual wavelet frames $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}\}_{j,k\in\mathbb{Z}}$ for which ψ and $\tilde{\psi}$ are finite linear combinations of functions $DT_kB_N, \ j,k \in \mathbb{Z}$.

The unitary extension principle

Solution: consider systems of the wavelet-type, but generated by more than one function.

Setup for construction of tight wavelet frames by Ron & Shen:

Let $\psi_0 \in L^2(\mathbb{R})$ and assume that

(i) There exists a function $H_0 \in L^\infty(\mathbb{T})$ such that

$$\widehat{\psi_0}(2\gamma) = H_0(\gamma)\widehat{\psi_0}(\gamma).$$

(ii) $\lim_{\gamma\to 0} \widehat{\psi_0}(\gamma) = 1.$

Further, let $H_1, \dots, H_n \in L^\infty(\mathbb{T})$, and define $\psi_1, \dots, \psi_n \in L^2(\mathbb{R})$ by

$$\widehat{\psi_\ell}(2\gamma) = H_\ell(\gamma)\widehat{\psi_0}(\gamma),\ \ell = 1, \dots, n.$$

The unitary extension principle

- $\widehat{\psi_0}(2\gamma) = H_0(\gamma)\widehat{\psi_0}(\gamma)$.
- $\widehat{\psi_\ell}(2\gamma) = H_\ell(\gamma)\widehat{\psi_0}(\gamma),\ \ell = 1, \dots, n$.
- We want to find conditions on the functions $H_1, \dots, H_n$ such that $\psi_1, \dots, \psi_n$ generate a tight multiwavelet frame for $L^2(\mathbb{R})$.
- Then

$$f = \sum_{\ell=1}^{n} \sum_{j,k\in\mathbb{Z}} \langle f, D^j T_k \psi_\ell \rangle D^j T_k \psi_\ell,\ \forall f \in L^2(\mathbb{R}).$$

- Let H denote the $(n+1) \times 2$ matrix-valued function defined by

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_n(\gamma) & T_{1/2}H_n(\gamma) \end{pmatrix},\ \gamma \in \mathbb{R}.$$

The unitary extension principle

Theorem (Ron and Shen, 1997): *Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be as in the general setup, and assume that $H(\gamma)^*H(\gamma) = I$ for a.e. $\gamma \in \mathbb{T}$. Then the multiwavelet system $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ constitutes a tight frame for $L^2(\mathbb{R})$ with frame bound equal to* 1.

The matrix $H(\gamma)^*H(\gamma)$ has four entries, but it is enough to verify two sets of equations:

Corollary: Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be as in the general setup and assume that

$$\sum_{\ell=0}^{n} |H_\ell(\gamma)|^2 = 1,$$

and

$$\sum_{\ell=0}^{n} \overline{H_\ell(\gamma)} T_{1/2} H_\ell(\gamma) = 0,$$

for a.e. $\gamma \in \mathbb{T}$. Then $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ constitutes a tight frame for $L^2(\mathbb{R})$ with frame bound equal to 1.

The unitary extension principle and B-splines

Exmple: For any $m = 1, 2, \ldots,$ we consider the (centered) B-spline

$$\psi_0 := B_{2m}$$

of order $2m$. Then

$$\widehat{\psi}_0(\gamma) = \left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)^{2m}.$$

It is clear that $\lim_{\gamma\to 0} \widehat{\psi}_0(\gamma) = 1$, and by direct calculation,

$$\widehat{\psi}_0(2\gamma) = \left(\frac{\sin(2\pi\gamma)}{2\pi\gamma}\right)^{2m} = \left(\frac{2\sin(\pi\gamma)\cos(\pi\gamma)}{2\pi\gamma}\right)^{2m} = \cos^{2m}(\pi\gamma)\widehat{\psi}_0(\gamma).$$

Thus ψ_0 satisfies a refinement equation with mask

$$H_0(\gamma) = \cos^{2m}(\pi\gamma).$$

The unitary extension principle and B-splines

Now, consider the binomial coefficient

$$\binom{2m}{\ell} := \frac{(2m)!}{(2m-\ell)!\ell!},$$

and define the functions $H_1, \ldots, H_{2m} \in L^\infty(\mathbb{T})$ by

$$H_\ell(\gamma) = \sqrt{\binom{2m}{\ell}} \sin^\ell(\pi\gamma) \cos^{2m-\ell}(\pi\gamma).$$

Direct calculation shows that $H(\gamma)^* H(\gamma) = I$.
Thus the $2m$ functions $\psi_1, \ldots, \psi_{2m}$ defined by

$$\begin{aligned} \widehat{\psi_\ell}(\gamma) &= H_\ell(\gamma/2)\widehat{\psi_0}(\gamma/2) \\ &= \sqrt{\binom{2m}{\ell}} \frac{\sin^{2m+\ell}(\pi\gamma/2)\cos^{2m-\ell}(\pi\gamma/2)}{(\pi\gamma/2)^{2m}} \end{aligned}$$

generate a tight frame $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,2m}$ for $L^2(\mathbb{R})$.

The unitary extension principle and B-splines

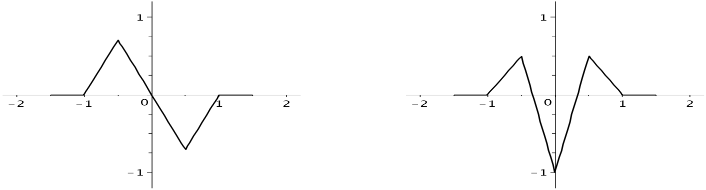

Figure: The two wavelet frame generators ψ_1 and ψ_2 associated with $\psi_0 = B_2$.

Shortcomings of the UEP

- The computational effort increases with the order of the B-spline B_{2m}: For higher orders, we need more generators, and more non-zero coefficients appear in ψ_ℓ.
- There is a limitation on the possible number of vanishing moments ψ_ℓ can have: in the B-spline case, at least one of the functions ψ_ℓ can only have one vanishing moment. This leads to sub-optimal approximation properties.

More recent extension principles, applications

- Mixed extension principle: construction of dual wavelet frames
- Oblique extension principle: equivalent to the UEP, but provides more natural constructions of frames with high approximation orders and optimal number of vanishing moments. Developed by

  Daubechies & Han & Ron & Shen, and Chui & He & Stöckler
- Pseudosplines by Daubechies & Han & Ron & Shen : based on the filter

$$H_0(\gamma) := \cos^{2m}\pi\gamma \sum_{k=0}^{\ell} \binom{m+\ell}{k} \sin^{2k}\pi\gamma \, \cos^{2(\ell-k)}\pi\gamma, \; \gamma \in \mathbb{R},$$

  where $\ell < m$ are nonnegative integers and the associated refinable function ψ_0 such that

$$\widehat{\psi_0}(2\gamma) = H_0(\gamma)\widehat{\psi_0}(\gamma).$$

More recent extension principles, applications

- Mixed oblique extension principle: dual frame variant of the OEP, but computationally much simpler (avoids spectral factorization). Yield decompositions

$$f = \sum_{\ell=1}^{n} \sum_{j,k \in \mathbb{Z}} \langle f, D^j T_k \widetilde{\psi_\ell} \rangle D^j T_k \psi_\ell, \forall f \in L^2(\mathbb{R}).$$

- The UEP is a special case of a much more general result in harmonic analysis that is not related to the wavelet structure (C. & Say Song Goh, forthcoming paper).

Wavelets and B-splines

Applications to image analysis (restoring, deblurring, inpainting) by Cai, Osher & Shen (2009-2015).

- Cai, J. F., Osher, S., and Shen, Z.: *Split Bregman methods and frame based image restoration.* Multiscale Model. Simul., **8** (2009), 337–369.
- Cai, J. F., Dong, B., Osher, S., and Shen, Z.: *Image restoration: Total variation, wavelet frames, and beyond.* J. Amer. Math. Soc. **25** (2012), 1033–1089.

Complex pseudosplines (C. & Forster & Massopust, 2015)

Consider the filter

$$H_0(\gamma) := (\cos^2 \pi\gamma)^z \sum_{k=0}^{\ell} \binom{z+\ell}{k} (\sin^2 \pi\gamma)^k (\cos^2 \pi\gamma)^{\ell-k}, \ \gamma \in \mathbb{R},$$

where $z \in \mathbb{C}$ with $\alpha := Re(z) \geq 1$ and $0 \leq \ell \leq \lfloor \alpha \rfloor - 1$, and

$$\binom{z+\ell}{k} := \frac{\Gamma(z+\ell+1)}{\Gamma(k+1)\Gamma(z+\ell-k+1)},$$

The filter H_0 generates a refinable function ϕ via the cascade algorithm, i.e.,

$$\widehat{\varphi}(\gamma) = \prod_{m=1}^{\infty} H_0(2^{-m}\gamma)\widehat{\varphi}(0), \quad \gamma \in \mathbb{R}.$$

Pseudosplines

Proposition Consider the filter H_0 and the associated refinable function φ. Furthermore, let

$$\eta(\gamma) := 1 - \left(|H_0(\gamma)|^2 + |H_0(\gamma + \tfrac{1}{2})|^2\right) \geq 0. \quad (1)$$

Let σ be a 1-periodic function such that $|\sigma(\gamma)|^2 = \eta(\gamma)$, and define the filters $\{H_n\}_{n=1}^3$ by

$$H_1(\gamma) = e^{2\pi i\gamma}\overline{H_0(\gamma + \tfrac{1}{2})}, \;\; H_2(\gamma) = \frac{1}{\sqrt{2}}\sigma(\gamma), \;\; H_3(\gamma) = \frac{1}{\sqrt{2}}e^{2\pi i\gamma}\sigma(\gamma).$$

Then the functions $\{\psi_n\}_{n=1}^3$ given by

$$\widehat{\psi_n}(2\gamma) = H_n(\gamma)\widehat{\varphi}(\gamma) \quad (2)$$

generate a tight frame $\{D^j T_k \psi_n\}_{j,k\in\mathbb{Z},n=1,2,3}$ with frame bound $A = 1$.

Advantages of complex pseudosplines

- Increased flexibility in regard to smoothness: instead of working with a discrete family of functions from C^m, $m \in \mathbb{N}_0$, we have a *continuous* family of functions belonging to the Hölder spaces $C^{\alpha-1}$.
- More reasons: B. Forster, Five Good Reasons for Complex-Valued Transforms in Image Processing.
    - Real-valued transforms can only provide a symmetric spectrum and are therefore unable to separate positive and negative frequency bands. Moreover, real-valued transforms are unusuable for all applications of phase retrieval, such as e.g. holography. Here, complex-valued transforms, bases and frames are indispensably needed.