Basic relations valid for the Bernstein space B_σ^2 and their extensions to functions from larger spaces in terms of their distances from B_σ^2

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The first lecture of this trilogy introduces the theorems and formulae from several areas of mathematical analysis, both for bandlimited functions belonging to a Bernstein space B_σ^2, and for functions from larger spaces of non-bandlimited functions, to be treated in detail. The second lecture presents a new, unified approach to theory and errors occurring when the results for B_σ^2 are extended to a function f from a larger space, in terms of the distance of f from B_σ^2. They also cover the difficult situation of derivative-free error estimates. The third lecture applies this new approach to the theorems of the first lecture but also treats Hilbert transforms as a further new application of the distance approach. The third lecture applies not only the new theoretical approach to the results of the first two lectures but also treats Hilbert transforms, in fact their higher order derivatives, as a further new application of the distance approach.

As to the first lecture, the four formulae for B_σ^2 presented in Section 1 are all equivalent to each other in the sense that each is a corollary of the others. Likewise the six formulae of Section 2 for the space F^2, the largest space in which the Fourier transform, our basic tool, can be applied effectively, are also equivalent to each other. What is surprising is that all nine formulae are even equivalent.

1 Basic theorems for bandlimited functions

Let B_σ^p for $\sigma > 0$, $1 \le p \le \infty$, be the Bernstein space of all entire functions $f\colon \mathbb{C} \to \mathbb{C}$ that belong to $L^p(\mathbb{R})$ when restricted to the real axis as well as are of exponential type σ, so that they satisfy the inequality $f(z) = \mathcal{O}_f\big(\exp(\sigma|\,\mathfrak{Im}\, z|\big)$ for $|z| \to \infty$. According to the Paley-Wiener theorem, the (distributional) Fourier transform of those functions has compact support contained in $[-\sigma, \sigma]$.

The classical sampling theorem of signal analysis, connected with the names of C. Shannon (1948/49), V. A. Kotelnikov (1933), E. T. Whittaker (1915), and many others, states that a function $f \in B_\sigma^2$ has the following representation:

Classical Sampling theorem (CST) For $f \in B^2_\sigma$ with some $\sigma > 0$ we have

$$f(z) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc} w\Big(z - \frac{k\pi}{\sigma}\Big) \quad (z \in \mathbb{C}), \tag{1}$$

convergence being absolute and uniform on compact subsets of $\mathbb{C}$.

The sinc-function is given by $\operatorname{sinc} z := \sin(\pi z)/(\pi z)$ for $z \neq 0$, and $:= 1$ for $z = 0$.

The first aim of this lecture is to show that this formula is equivalent to several other striking formulae of mathematical analysis such as:

Poisson's summation formula (PSF particular case) For $f \in B^1_\sigma$:

$$\int_{\mathbb{R}} f(t)\, dt = \frac{2\pi}{\sigma} \sum_{k\in\mathbb{Z}} f\Big(\frac{2k\pi}{\sigma}\Big). \tag{2}$$

General Parseval formula (GPF) For $f, g \in B^2_\sigma$ with $\sigma > 0$:

$$\int_{\mathbb{R}} f(t)\, \overline{g}(t)\, dt = \frac{\pi}{\sigma} \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\, \overline{g}\Big(\frac{k\pi}{\sigma}\Big). \tag{3}$$

Reproducing kernel formula (RKF) For $f \in B^2_\sigma$ with $\sigma > 0$:

$$f(z) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(t) \operatorname{sinc}\Big(\frac{\sigma}{\pi}(z - t)\Big)\, dt \qquad (z \in \mathbb{C}). \tag{4}$$

This means that B^2_σ is a reproducing kernel Hilbert space, i. e., there exists a kernel function $k(\cdot, z)$ which belongs to B^2_σ for each $z \in \mathbb{C}$, such that

$$f(z) = \big\langle f(\cdot), k(\cdot, z)\big\rangle \qquad (z \in \mathbb{C}).$$

Valiron's or Tschakaloff's sampling/interpolation formula (VSF) For $f \in B^\infty_\sigma$ with $\sigma > 0$ we have for all $z \in \mathbb{C}$:

$$f(z) = \big(f'(0)z + f(0)\big) \operatorname{sinc}\Big(\frac{\sigma z}{\pi}\Big) + \sum_{k\in\mathbb{Z}\setminus\{0\}} f\Big(\frac{k\pi}{\sigma}\Big) \frac{\sigma z}{k\pi} \operatorname{sinc}\Big(\frac{\sigma z}{\pi} - k\Big), \tag{5}$$

the convergence being absolute and uniform on compact subsets of $\mathbb{C}$.

2 Their extensions to non-bandlimited functions

We now weaken the assumption of $f \in B^2_\sigma$, i. e., the Fourier transform $\widehat{f}$ has support contained in $[-\sigma, \sigma]$, to $\widehat{f} \in L^1(\mathbb{R})$. In this respect we introduce the *Fourier inversion class*

$$F^p := \Big\{ f \colon \mathbb{R} \to \mathbb{C} \;:\; f \in L^p(\mathbb{R}) \cap C(\mathbb{R}),\ \widehat{f} \in L^1(\mathbb{R}) \Big\},$$

as well as the ℓ^p *summability class* for *step size* $h > 0$

$$S_h^p := \Big\{ f : \mathbb{R} \to \mathbb{C} \ : \ \big(f(hk)\big)_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z}) \Big\}.$$

In the frame of these spaces all the formulae mentioned above hold only approximately in the sense that they have to be equipped with remainder (additional) terms. More precisely, the classical sampling theorem (1) is replaced by the

Approximate/extended sampling theorem (AST) For $f \in F^2 \cap S^1_{\pi/\sigma}$:

$$f(t) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc} \frac{\sigma}{\pi}\Big(t - \frac{k\pi}{\sigma}\Big) + (R_\sigma f)(t) \qquad (t \in \mathbb{R}), \tag{6}$$

$$(R_\sigma f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \Big(1 - e^{-i2k\sigma t}\Big) \int_{(2k-1)\sigma}^{(2k+1)\sigma} \widehat{f}(v) e^{ivt}\, dv. \tag{7}$$

the series converging absolutely and uniformly on $\mathbb{R}$. Moreover, the remainder $R_\sigma f$ can be estimated by

$$\big|(R_\sigma f)(t)\big| \le \sqrt{\frac{2}{\pi}} \int_{|v|\ge\sigma} |\widehat{f}(u)|\, du = o(1) \quad (\sigma \to \infty), \tag{8}$$

which yields

$$\lim_{\sigma\to\infty} \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc} \frac{\sigma}{\pi}\Big(t - \frac{k\pi}{\sigma}\Big) = f(t) \quad \text{(uniformly for } t \in \mathbb{R}).$$

The particular case of Poisson's summation formula for $f \in B^1_\sigma$, thus (2), is generalized to the classical form:

Poisson's summation formula (PSF) for $f \in F^1$ with $\widehat{f} \in S^1_{\pi/\sigma}$:

$$\sqrt{2\pi}\frac{\sigma}{\pi} \sum_{k\in\mathbb{Z}} f\Big(x + \frac{2k\sigma}{\pi}\Big) = \sum_{k\in\mathbb{Z}} \widehat{f}\Big(\frac{k\pi}{\sigma}\Big) e^{ik\pi x/\sigma} \quad (a.\,e.). \tag{9}$$

In case of the general Parseval formula (3) one has even to add two remainder terms, leading to

Generalized Parseval decomposition formula (GPDF) For $f \in F^2 \cap S^1_{\pi/\sigma}$, $\sigma > 0$,and $g \in F^2$, there holds $R_\sigma f \in L^2(\mathbb{R})$ and

$$\begin{aligned} \int_{\mathbb{R}} f(u)\overline{g}(u)\, du = \frac{\pi}{\sigma} \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\overline{g}\Big(\frac{k\pi}{\sigma}\Big) \\ - \frac{\pi}{\sigma}\frac{1}{w} \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \frac{1}{\sqrt{2\pi}} \int_{|v|\ge\sigma} \widehat{\overline{g}}(v) e^{ik\pi v/\sigma}\, dv \\ + \int_{\mathbb{R}} (R_\sigma f)(u)\overline{g}(u)\, du, \end{aligned} \tag{10}$$

where $R_\sigma f$ is given by (7). Observe that in view of (8), $\lim_{\sigma\to\infty}(R_\sigma f)(t) = 0$ uniformly for $t \in \mathbb{R}$.

Similarly, the reproducing kernel formula (4) has to be equipped with an additional term:

Approximate reproducing kernel formula (ARKF) For $f \in F^2$:

$$f(t) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \operatorname{sinc}\Big(\frac{\sigma}{\pi}(t-u)\Big)\, du + (R_\sigma^* f)(t) \tag{11}$$

with

$$|(R_\sigma^* f)(t)| := \left| \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} \widehat{f}(v) e^{itv} dv \right| \le \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} |\widehat{f}(v)| dv = o(1) \quad (\sigma \to \infty).$$

Clearly, if the functions involved belong to the (particular) Bernstein spaces B_σ^2, then, according to the Paley-Wiener theorem, the remainder terms in (8), (10) and (11) vanish, and one obtains the particular versions (1), (3) and (4). Similarly, for $f \in B_\sigma^1$ and $x = 0$, Poisson's summation formula (9) reduces to the particular case (2).

3 Boas-type formulae for higher derivatives

The basis to well known Bernstein inequality for functions $f \in B_\sigma^2$, namely, $\|f^{(s)}\|_{L^2(\mathbb{R})} \le \sigma^s \|f\|_{L^2(\mathbb{R})}$, is the following formula of Boas:

Let $f \in B_\sigma^\infty$, where $\sigma > 0$. Then, for $h = \pi/\sigma$, we have

$$f'(t) \;= \frac{1}{h} \sum_{k\in\mathbb{Z}} \frac{(-1)^{k+1}}{\pi(k-\frac{1}{2})^2}\, f\Big(t + h\Big(k - \frac{1}{2}\Big)\Big)\,.$$

Its extension to higher order derivatives is given by (see [8]):

Theorem 3.1. *Let $f \in B_\sigma^\infty$ for some $\sigma > 0$. Then for $s \in \mathbb{N}$, and $h := \pi/\sigma$,*

$$f^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k=-\infty}^{\infty} (-1)^{k+1} A_{s,k}\, f\Big(t + h\Big(k - \frac{1}{2}\Big)\Big) \qquad (t \in \mathbb{R}), \tag{12}$$

where

$$A_{s,k} := \frac{(2s-1)!}{\pi(k-\frac{1}{2})^{2s}} \sum_{j=0}^{s-1} \frac{(-1)^j}{(2j)!} \big[\pi\big(k - \tfrac{1}{2}\big)\big]^{2j} \qquad (k \in \mathbb{Z})$$

The new extension to non-bandlimited functions reads:

Theorem 3.2. *Let $s \in \mathbb{N}$, $f \in F^2$ and let $v^{2s-1}f(v)$ be absolutely integrable. Then $f^{(2s-1)}$ exists and for $h > 0$, $\sigma := \pi/h$ formula* (12) *extends to*

$$f^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{s,k} f\Big(t + h\Big(k - \frac{1}{2}\Big)\Big) + (R_{2s-1,\sigma} f)(t)\,,$$

where

$$(R_{2s-1,\sigma} f)(t) = \frac{i(-1)^{s-1}}{\sqrt{2\pi}\, h^{2s-1}} \int_{|v|\ge\sigma} \big[(hv)^{2s-1} - \phi_{2s-1}(hv)\big]\, \widehat{f}(v) e^{ivt}\, dv$$

with ϕ_{2s-1} being the 4π-periodic function defined by

$$\phi(v) = \begin{cases} v^{2s-1}, & -\pi \le v \le \pi\,, \\ (2\pi - v)^{2s-1}, & \pi < v \le 3\pi\,. \end{cases}$$

In particular,

$$\big|(R_{2s-1,\sigma} f)(t)\big| \le \sqrt{\frac{2}{\pi}} \int_{|v|\ge\sigma} |v|^{2s-1} \big|\widehat{f}(v)\big|\, dv = o(1)\ (\sigma \to \infty).$$

Furthermore, there holds the extended Bernstein-type inequality,

$$\|f^{(2s-1)}\|_{L^2(\mathbb{R})} \le \sigma^{2s-1} \|f\|_{L^2(\mathbb{R})} + \|R_{2s-1,\sigma} f\|_{L^2(\mathbb{R})}$$

with

$$\big\|R_{2s-1,\sigma} f\big\|_{L^2(\mathbb{R})} \le 2 \bigg\{ \int_{|v|\ge\sigma} \big|v^{2s-1} \widehat{f}(v)\big|^2\, dv \bigg\}^{1/2} = o(1)\ (\sigma \to \infty).$$

Similar results hold for even order derivatives.

4 Foundations for a unified approach to extensions: A hierarchy of wider spaces and estimates for the distance of f from B^2_σ

We have seen that there exist formulae for functions in B^p_σ that hold for $f \in F^p$ (or a subspace of it) with a remainder $R_\sigma f$ tending to zero as $\sigma \to \infty$. Now we aim at a unified approach to such extensions with error estimates in terms of the distance of f from B^p_σ.

Motivation. There exist numerous relations (equations or inequalities) of the form

$$U(f) = V_\sigma(f) \quad \text{or} \quad U(f) \le V_\sigma(f) \qquad (f \in B^2_\sigma),$$

where U and V_σ are functionals; see [1].

An example of an equation is CFT where $U(f) = f(z)$ and $V_\sigma(f)$ is the sampling series; see (1). An example of an inequality is Bernstein's inequality in $L^2(\mathbb{R})$, where $U(f) = \|f'\|_{L^2(\mathbb{R})}$ and $V_\sigma(f) = \sigma\|f\|_{L^2(\mathbb{R})}$. These relations are no longer valid outside B^2_σ. But if f is in some sense close to B^2_σ, then these relations will not fail drastically. They will hold with a remainder $R_\sigma f$ so that

$$U(f) = V_\sigma(f) + R_\sigma f \quad \text{or} \quad U(f) \le V_\sigma(f) + R_\sigma f \qquad (f \in B^2_\sigma)$$

with $R_\sigma f$ depending on the distance of f from B^2_σ.

A hierarchy of spaces. In our approach, the Fourier inversion class F^p for $p \in [1,2]$ is the largest superior space of B^p_σ in which a representation of the remainder $R_\sigma f$ can be guaranteed. However, if we want $R_\sigma f$ to converge rapidly to zero as $\sigma \to \infty$, we should rather consider a subspace of F^p. It is therefore desirable to know a hierarchy of spaces lying between B^p_σ and F^p. Our considerations include:

- the *modulation space* $M^{2,1}$;
- a subspace $M^{2,1}_*$ of $M^{2,1}$ created by a uniform dilation process;
- the *Lipschitz space* $\mathrm{Lip}_r(\alpha, L^2(\mathbb{R}))$;
- the *Sobolev space* $W^{r,p}(\mathbb{R})$;
- the *Hardy space* $H^p(\mathcal{S}_d)$ of functions f analytic in the strip $\mathcal{S}_d := \{z \in \mathbb{C} : |\Im z| < d\}$.

For these spaces, we show the following inclusions

$$B^p_\sigma|_\mathbb{R} \subsetneqq H^p(\mathcal{S}_d)|_\mathbb{R} \subsetneqq W^{r,p}(\mathbb{R}) \cap C(\mathbb{R}) \subsetneqq F^p \cap S^p_h \subsetneqq F^p \subsetneqq L^p(\mathbb{R}).$$

Here $|_\mathbb{R}$ means that the functions of the corresponding space are restricted to $\mathbb{R}$. For $p = 2$ these inclusions can be further refined by involing the Lipschitz and the modulation spaces. We have

$$W^{r,2}(\mathbb{R}) \cap C(\mathbb{R}) \subsetneqq M^{2,1}_* \subsetneqq M^{2,1} \subsetneqq F^2 \cap S^2_h$$

and

$$M^{2,1}_* \subsetneqq \mathrm{Lip}_r(\tfrac{1}{2}, L^2(\mathbb{R})) \cap F^2.$$

Norms and distances. In order to measure the distance of a function f belonging to F^2 (or to one of its subspaces) from B^2_σ, we need to introduce a metric in F^2. For $q \in [1,2]$ and $f \in F^2$, we define

$$\| f \|_q := \left\{ \int_{\mathbb{R}} |\widehat{f}(v)|^q \, dv \right\}^{1/q} \equiv \|\widehat{f}\|_{L^q(\mathbb{R})},$$

which endowes F^2 with a norm. It induces a metric which allows us to define the distance of f from B^2_σ as

$$\mathrm{dist}_q(f, B^2_\sigma) := \inf_{g \in B^2_\sigma} \| f - g \|_q \equiv \inf_{g \in B^2_\sigma} \|\widehat{f} - \widehat{g}\|_{L^q(\mathbb{R})},$$

For this notion of distance, the following two fundamental results hold:

$$\mathrm{dist}_q(f, B^2_\sigma) = \left\{ \int_{|v|>\sigma} \big|\widehat{f}(v)\big|^q dv \right\}^{1/q} \le c \left\{ \int_\sigma^\infty v^{-q/2} \big[\omega_r(f, v^{-1}, L^2(\mathbb{R}))\big]^q dv \right\}^{1/q},$$

and if in addition $v\widehat{f}(v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then f' exists and

$$\mathrm{dist}_q(f', B^2_\sigma) = \left\{ \int_{|v|>\sigma} \big|v\widehat{f}(v)\big|^q dv \right\}^{1/q} \le c \left\{ \int_\sigma^\infty v^{-q/2} \big[\omega_r(f', v^{-1}, L^2(\mathbb{R}))\big]^q dv \right\}^{1/q}.$$

Here $\omega_r(\cdot, \cdot, L^2(\mathbb{R}))$ denotes the modulus of smoothness of order r in $L^2(\mathbb{R})$.

For the subspaces of F^2 estimates of the distances and rates of convergence as $\sigma \to \infty$ will be determined. For example, if $f \in M^{2,1}_*$, then

$$\mathrm{dist}_q(f, B^2_\sigma) = \mathcal{O}(\sigma^{-1+1/q}) \quad (q \in (1,2]),$$

and if $f \in \mathrm{Lip}_r(\beta, L^2(\mathbb{R})) \cap F^2$ and $1/q - 1/2 < \beta \le r$, then

$$\mathrm{dist}_q(f, B^2_\sigma) = \mathcal{O}(\sigma^{-\beta-1/2+1/q}).$$

In Sobolev spaces the distances also converge to zero like a power of $1/\sigma$ and in Hardy spaces they converge to zero exponentially.

5 Applications of the distance approach to the remainders of the formulae of our trilogy

After these preparations we turn to the errors involved under the extensions to larger spaces. The new results include:

- the sampling formula of Whittaker–Kotel'nikov–Shannon [6, 7] (see below);

- the sampling formula of Tschakaloff-Valiron [4];
- the general Parseval formula [2, 3];
- the reproducing kernel formula [4];
- the differentiation formula of Boas [4, 5, 8];
- Bernstein's inequality [4];
- Nikol'skiĭ's inequality [4];
- the counterpart of Boas's differentiation formula for the Hilbert transform [5].

In these cases, we investigate the error terms in the formulae of Section 2 in terms of the distance from B_σ^2. As one concrete example for the remainder in the extended sampling theorem (6) we have. In this case we have the *derivative free* estimate in terms of the rth order modulus of smoothness,

$$|(R_\sigma f)(t)| \le \sqrt{\frac{2}{\pi}} \int_{|v|\ge\sigma} |\widehat{f}(v)|\, dv = \sqrt{\frac{2}{\pi}}\, \mathrm{dist}_1(f, B_\sigma^2)$$

$$\le c\left\{ \int_\sigma^\infty v^{-1}\omega_r(f, v^{-1}, L^2(\mathbb{R}))\, dv \right\} \qquad (\sigma > 0).$$

Thus $U(f) = f(t)$, $V_\sigma(f) = (S_\sigma f)(t)$, with $U(f) = V_\sigma(f) + (R_\sigma f)(t)$. If $f \in \mathrm{Lip}_r(\alpha, L^2(\mathbb{R})) \cap C(\mathbb{R}) \cap S_h^2$, $1/2 < \alpha \le r$, then $(R_\sigma f)(t) = \mathcal{O}(\sigma^{1/2-\alpha})$, $\sigma \to \infty$. If $f \in W^{r,2}(\mathbb{R})$, then $|(R_\sigma f)(t)| \le c\,\sigma^{1/2-r}\|f^{(r)}\|_{L^2(\mathbb{R})}$, and $f \in H^2(\mathcal{S}_d)$ implies $|(R_\sigma f)(t)| \le c\, \exp(-d\sigma)\|f\|_{H^2(\mathcal{S}_d)}$.

References

[1] P. L. Butzer, P. J. S. G. Ferreira, J. R. Higgins, G. Schmeisser, and R. L. Stens. The sampling theorem, Poisson's summation formula, general Parseval formula, reproducing kernel formula and the Paley-Wiener theorem for bandlimited signals - their interconnections. *Applicable Analysis*, 90(3-4):431–461, 2011.

[2] P. L. Butzer, P. J. S. G. Ferreira, J. R. Higgins, G. Schmeisser, and R. L. Stens. The generalized Parseval decomposition formula, the approximate sampling theorem, the approximate reproducing kernel formula, Poisson's summation formula and Riemann's zeta function; their interconnections for non-bandlimited functions. *to appear.*

[3] P. L. Butzer and A. Gessinger. The approximate sampling theorem, Poisson's sum formula, a decomposition theorem for Parseval's equation and their interconnections. *Ann. Numer. Math.*, 4(1-4):143–160, 1997.

[4] P. L. Butzer, G. Schmeisser, and R. L. Stens. Extensions of basic relations valid for bernstein spaces to wider spaces. *to appear.*

[5] P. L. Butzer, G. Schmeisser, and R. L. Stens. Boas-type formulae for higher derivatives, their extension to non-bandlimited functions and applications. *to appear.*

[6] P. L. Butzer and W. Splettstösser. A sampling theorem for duration limited functions with error estimates. *Inf. Control*, 34:55–65, 1977.

[7] P. L. Butzer, W. Splettstösser, and R. L. Stens. The sampling theorem and linear prediction in signal analysis. *Jber.d.Dt.Math.-Verein.*, 90:1–70, 1988.

[8] G. Schmeisser. Numerical differentiation inspired by a formula of R. P. Boas. *J. Approximation Theory*, 160:202–222, 2009.