# On 1-Planar Graphs with Rotation Systems

Christopher Auer, Franz. J. Brandenburg, Andreas Gleißner, Josef Reislhuber

Department of Informatics and Mathematics, University of Passau
{auerc,brandenb,gleissner,reislhuber}@fim.uni-passau.de

UNIVERSITÄT PASSAU
Fakultät für Mathematik und Informatik

Technical Report, Number MIP-1207
Department of Informatics and Mathematics
University of Passau, Germany
September 2012

# On 1-Planar Graphs with Rotation Systems

**Technical Report MIP-1207**
**September, 2012**

Christopher Auer, Franz J. Brandenburg, Andreas Gleißner, and Josef Reislhuber

University of Passau
94030 Passau, Germany,
{auerc,brandenb,gleissner,reislhuber}@fim.uni-passau.de

**Abstract.** A graph is 1-planar if it can be drawn in the plane such that each edge is crossed at most once. This causes essential distinctions to planar graphs: planarity can be tested in linear time whereas 1-planarity is **NP**-hard [11]. We improve this result and show the **NP**-hardness for 1-planar graphs with a given rotation system. In addition, the crossing number problem remains **NP**-hard for 1-planar graphs even with a rotation system. However, there are tractable cases: 1-planarity can be tested efficiently for embedded graphs and for maximal graphs with a given rotation system.

## 1 Introduction

Planar graphs have attracted researchers since the 1930's. There are numerous results on planar graphs such as forbidden minors, duality, efficient planarity tests and straight line drawings, see [1, 10, 15]. More recently, researchers have investigated graphs that are "almost" planar; here the number of edges is linearly bounded by the number of vertices [16]. Almost planar graphs admit crossings in some controlled way, such that the linear density is preserved. A particular example is 1-planarity. 1-planar graphs were introduced by Ringel [18] in an approach to color a planar graph and its dual. 1-planar graphs are not yet fully explored. A 1-planar graph with $n$ vertices has at most $4n - 8$ edges and this bound is tight [5, 16]. 1-planar graphs do not admit straight-line drawings [4]. They are not closed under edge contraction and there are infinitely many minimal non-1-planar graphs [12]. Recently, Korzhik and Mohar [11] proved that deciding whether a given graph $G$ is 1-planar is **NP**-hard by a sophisticated reduction from the 3-colorability problem of planar graphs of degree at most four.

A rotation system describes the cyclic ordering of the edges at the vertices as obtained from a drawing. Planarity tests commonly output a rotation system, which is used to compute planar embeddings and straight-line planar drawings in linear time [1, 10, 15]. Rotation systems play a crucial role in this work.

The rotation system makes the essential difference for the complexity of upward planarity testing. A directed graph is upward planar if it can be drawn in the plane such that the curves of the edges are monotonically increasing in $y$-direction. Garg and Tamassia [8] showed that upward planarity testing of a graph is **NP**-hard. However, there is a linear time algorithm if a rotation system is given [1, 2]. In contrast, the **NP**-hard crossing number problem [7] remains **NP**-hard even with a given rotation system [17]. There is a parallel situation for 1-planarity.

We show that 1-planarity testing remains **NP**-hard with a given rotation system. Our **NP**-reduction is general enough to hold without a rotation system as well, and it can be modified to show that the crossing number problem remains **NP**-hard even for 1-planar graphs. Our proof is by a reduction from the planar 3-SAT problem [13] and is an alternative to the one by Korzhik and Mohar [11].

On the other hand, 1-planarity can be tested in linear time if the embedding is given. Moreover, we show that there is an efficient test whether a graph with a given rotation system is maximal 1-planar. In a maximal 1-planar graph any further edge violates 1-planarity. They are of interest in their own rights, since their density ranges between $\frac{45}{17}n$ and $4n-8$ edges and between $\frac{7}{3}n$ and $4n-8$ edges if the rotation system is fixed, as we have explored in our companion paper on 1-planarity [3].

## 2 Preliminaries

We consider simple undirected graphs $G = (V, E)$ with $n$ vertices and $m$ edges. A *drawing* of a graph is a mapping of $G$ into the plane such that the vertices are mapped to distinct points and each edge is a Jordan arc between its endpoints. A drawing is *plane* if the Jordan arcs of the edges do not cross and it is *1-plane* if each edge is crossed at most once. In 1-plane drawings, crossings of edges with the same endpoint are excluded.

Each plane (1-plane) drawing of a graph implies a *rotation system*. The rotation at a vertex is the clockwise order of its incident edges as implied by the drawing. A rotation system is the list of rotations of all vertices. Note that in general a given rotation system of a graph may not allow for a plane (1-plane) drawing. Hence, we call a rotation system planar (1-planar) if it admits a plane (1-plane) drawing.

A rotation system of a graph defines an embedding of a graph on an orientable surface, [14]. However, in the plane it may be different from a topological *embedding*. These terms can be identified for planar graphs, since one can be computed from the other in linear time. A topological embedding can be obtained from a plane drawing. It specifies the faces, where a face is given by a cyclic sequence of the edges which forms its boundary.

Similar to planar embeddings, a *1-planar embedding* specifies the faces in a 1-planar drawing. A face in a 1-planar embedding is given by a cyclic list of edges and edge segments, where the latter occurs in the case of a crossing; see Fig. 9(b) for an example. Hence, in 1-planar embeddings, an edge may occur in up to four faces. As with planar graphs, a 1-planar embedding uniquely implies a 1-planar rotation system. However, a 1-planar rotation system does not uniquely define a 1-planar embedding nor the edges that cross. In fact, we shall use this "gap" to show that deciding whether a rotation system is 1-planar is **NP**-hard.

Let $G$ be a 1-planar embedded graph and denote by $G^\times$ its *planarization*. $G^\times$ is obtained from $G$ by replacing each pair $e = \{u, v\}$ and $e' = \{u', v'\}$ of crossing edges by a new vertex of degree four joined to $u$, $v$, $u'$, and $v'$. Then, $G^\times$ is a planar embedded graph, where its embedding is inherited from $G$.

A planar graph has a unique planar embedding if it is tri-connected. Similarly, a 1-planar graph has a *unique 1-planar embedding* if the following conditions hold [11]: if two edges $e$ and $e'$ of $G$ cross in any 1-planar embedding, then they cross in every 1-planar embedding and the planarization $G^\times$ of $G$ has a unique planar embedding. In addition, the uniqueness of a 1-planar embedding has been established in [3] if the graphs are maximal planar with a given rotation system. A graph is maximal if no further edge can be added without violating its defining property.

## 3 NP-hardness of 1-Planarity Testing

In this section, we reduce planar 3-SAT to 1-planarity using gadgets for literals, variables and clauses. The key idea is to encode truth values of variables by crossings. An edge connecting a literal with a clause is crossed if the literal is assigned false. One basic building block of our reduction is the *U-graph* which we adopt from [11]; see Fig. 1 for an example. The vertices labeled $3, 2, 1, b, b-1, b-2$ are called *boundary vertices* and an edge connecting two boundary vertices is called *boundary edge*. Korzhik and Mohar [11] proved that a U-graph has a unique 1-planar embedding if it has at least $b \geq 6$ boundary vertices. In our reduction, we attach barrier edges and gadgets for variables (*V-gadgets*) and clauses (*C-gadgets*) to the boundary vertices and we assume that the number of boundary vertices is always at least 6 and sufficiently large for the case at hand.

Let $G$ be the planar, embedded graph corresponding to a planar 3-SAT expression $\alpha$. In the following, we construct a graph $G^*_S$ endowed with a rotation system that is 1-planar if and only if $\alpha$ is satisfiable (see Fig. 2 for an example). The rotation system can be obtained directly from the given drawings. Let $G^*$ be the dual graph of $G$. First, we transform $G^*$ into a *U-supergraph* $G^*_S$ as described in [12]. The construction replaces every vertex of $G^*$ with a U-graph. Two adjacent vertices of $G^*$ are connected in $G^*_S$ by a set of $l$ edges, called *barrier*, where we choose $l \geq 7$ for reasons which will be described later. A U-supergraph has a unique 1-planar embedding [11]. For every vertex $v$ of $G$ that represents a clause (variable), we add a C-gadget (V-gadget) to $G^*_S$. Let $v$ be a vertex of $G$, $f$ be the corresponding face in $G^*$, and $F'$ be the set of vertices of $G^*$ at the boundary of $f$. In $G^*_S$, the vertices in $F'$ are replaced by U-graphs $F'_U$. In our construction, we "attach" each C- or V-gadget to an arbitrary U-graph in $F'_U$ such that the gadget lies inside the face of $G^*_S$ that corresponds to $v$ in $G$. Fig. 2(b) shows an example in which V-gadget $X_1$ is attached to U-graph $U_{f_1}$. Finally, for every edge between a clause and a variable vertex in $G$, we add a path, called *rope*, between the corresponding C- and V-gadgets in $G^*_S$. For the number of edges of

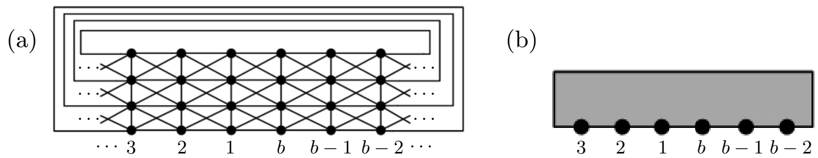


**Fig. 1.** The U-graph (a) and its abbreviation (b).

a rope, we choose two more edges than the number of edges of a barrier. As we will see later, a rope acts as a communication line that "passes" a crossing at a V-gadget to the C-gadgets at its other end. By our construction, we essentially obtain a simultaneous embedding of $G$ and its dual $G^*$ by means of our gadgets and the U-supergraph, respectively.

A simple example for $G_S^*$ is given in Fig. 2. The graph is obtained from a planar 3-SAT instance consisting of two clauses $C_1, C_2$ and three variables $X_1, X_2, X_3$ with the corresponding planar graph $G$; see Fig. 2(a). The vertices of $G$ are depicted as circles and the edges as straight-line segments, the vertices of the dual graph $G^*$ as squares and the edges as curled lines. Fig. 2(b) shows $G_S^*$, which is obtained from $G$ and $G^*$. The shaded rectangles represent the U-graphs, which are connected by the barriers, drawn as a bundle of lines. The semi-ellipses are the C- and V-gadgets with the according labels. The ropes are depicted as dashed lines. For the following argumentation, we need that no boundary edge is crossed.

**Lemma 1.** *In a 1-planar drawing of $G_S^*$ respecting the given rotation system, a boundary edge, which is an edge between two boundary vertices of a U-graph, is never crossed.*

Since we need the structure of the C- and V-gadgets in order to prove Lemma 1, we postpone the proof until all the necessary definitions are made.

First we consider C-gadgets used for the clauses; see Fig. 3 for an example. The gadget is attached to boundary vertices $b_1, \ldots, b_6$ of a U-graph. These vertices form the *clause base*. The vertices $v_1, v_2, v_3$ are the *variable vertices*, where each vertex corresponds to a literal in the clause. Hence, there are always three variable vertices. A variable vertex is connected to two vertices of the clause base by *anchor edges*. Additionally, a variable vertex is connected to the corresponding V-gadget via a rope. The edge from a variable vertex to the rope is called *variable edge* ($\{v_i, t_i\}$ for $i = 1, 2, 3$ in Fig. 3(a)). We introduce a path from $b_1$ to $b_6$, called *membrane*, which consists of the *membrane vertices* $m_1, \ldots, m_4$ connected by *membrane edges*.

In the following lemma, we need that the rope crosses at least one edge of a V-gadget. By Lemma 4, we will show that this precondition is always fulfilled.

**Lemma 2.** *In every 1-planar drawing of $G_S^*$ respecting the given rotation system, at least one incident edge of each vertex $v_1, v_2, v_3$ of a C-gadget is crossed by a membrane edge if the rope crosses at least one edge of the attached V-gadget.*

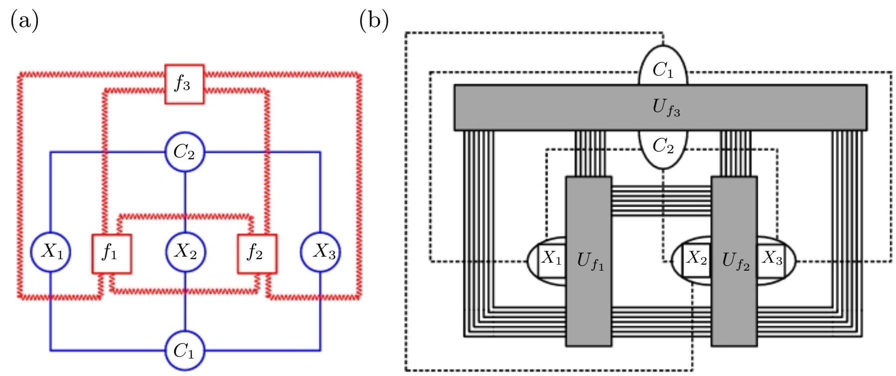


**Fig. 2.** Example for the 1-planar graph constructed in the reduction. (a) The plane drawing of a planar 3-SAT expression and its dual graph. (b) The corresponding U-supergraph $G_S^*$ with the clause and variable gadgets

*Proof.* W. l. o. g., we consider $v_1$. The first possibility to avoid a crossing of an adjacent edge of $v_1$ with a membrane edge is to cross a different edge of the rope other than $\{v_1, t_1\}$. A rope connects the C-gadget with a V-gadget with a barrier in between. In a 1-planar drawing, every edge of the rope must cross an edge of the barrier and an edge of the attached V-gadget by assumption. The size of a rope is the size of a barrier plus two. Hence, there is only one rope edge left to cross, namely $\{v_1, t_1\}$. Thus, to avoid a crossing of $v_1$'s edges with the membrane, the membrane needs to be drawn "around" the whole rope of $v_1$, i. e., the face enclosed by the membrane and the clause base must include the whole rope. Then, the membrane, consisting of five edges, must be routed through at least one barrier, consisting of at least seven edges, which is impossible. □

From the proof of Lemma 2 we obtain that there are only two possibilities for a variable vertex $v$: (A) Both anchor edges of $v$ are crossed by a membrane edge and its variable edge is not crossed by a membrane edge. (B) The variable edge of $v$ is crossed by a membrane edge and none of its two anchor edges is crossed by a membrane edge. If (B) holds, we say that a variable vertex lies *inside*, as seen in Fig. 3(a), where vertices $v_1$ and $v_2$ lie inside. If (A) holds, a variable vertex lies *outside*, e.g., vertex $v_3$ lies outside in Fig. 3(a). As a direct consequence of (A) and (B), in a 1-planar drawing it is not possible that all three variable vertices of a C-gadget lie outside at the same time, as this would require six membrane edges to be crossed. We exploit this property to encode if a clause represented by the C-gadget is satisfied, i. e., it is satisfied if and only if at least one variable vertex lies inside, which holds if and only if a 1-planar drawing is possible.

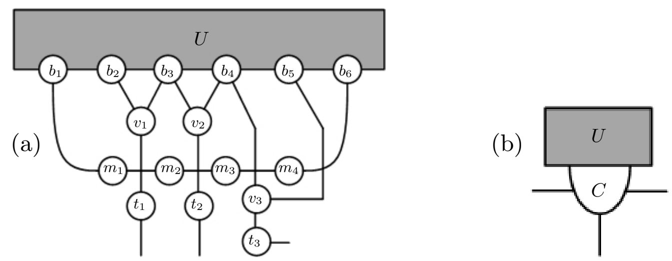


**Fig. 3.** A clause gadget (a) and its abbreviation (b).

The V-gadget for a variable consists of several *literal gadgets (L-gadgets)*. L-gadgets are similar to C-gadgets and come in two flavors, namely, a positive and a negative version; Fig. 5(a) depicts a positive L-gadget. The truth value of a single literal is encoded by a crossing of a certain edge of an L-gadget. An L-gadget has a *positive literal vertex* $l^+$ and *negative literal vertex* $l^-$ that are connected to three and two boundary vertices, respectively, of a U-graph via anchor edges. Vertex $l^+$ is connected to a rope vertex via a *clause edge* (named "Clause" in Fig. 5(a)). Additionally, $l^+$ is adjacent to the negative literal vertex of a neighboring L-gadget of the same V-gadget ("In" in Fig. 5(a)). Similarly, vertex $l^-$ is connected to the positive literal vertex of another neighboring L-gadget ("Out" in Fig. 5(a)). As in C-gadgets, in an L-gadget the boundary vertices $b_1$ and $b_7$ are connected by a membrane, consisting of the membrane vertices $m_1, m_2, m_3$. In a negative L-gadget the clause edge is incident to the negative literal vertex $l^-$ instead of the positive literal vertex $l^+$. Intuitively, the clause edge propagates the truth assignment of the literal via a rope to the clause in which the literal occurs, i. e., if the edge crosses the membrane, the literal is assigned false; true otherwise. The edges marked "In" and "Out" propagate the truth assignment of the literal to the other L-gadgets of the same variable to ensure a consistent truth value, i. e., either all positive or all negative L-gadgets cross their membranes. To ensure a consistent truth value, we additionally need the *terminal L-gadget* which is an L-gadget with no connection to a rope. The terminal L-gadgets are placed at the beginning and end of a series of L-gadgets; hence, the name.

As in Lemma 2, in Lemma 3, we again need that the rope crosses at least one edge of a V-gadget.

**Lemma 3.** *In every 1-planar drawing of $G_S^*$ respecting the given rotation system, at least one incident edge of each vertex $l^+, l^-$ of an L-gadget is crossed by a membrane edge if the rope crosses at least one edge of the respective V-gadget.*

*Proof.* The proof for vertex $l^+$ is analogous to the proof of Lemma 2. As a result of this, one edge of the membrane is already crossed, which leaves three membrane edges to consider. Vertex $l^-$ is adjacent to the $l^+$ vertex of another L-gadget now to referred as $l^+$. In order to avoid a crossing between any edge

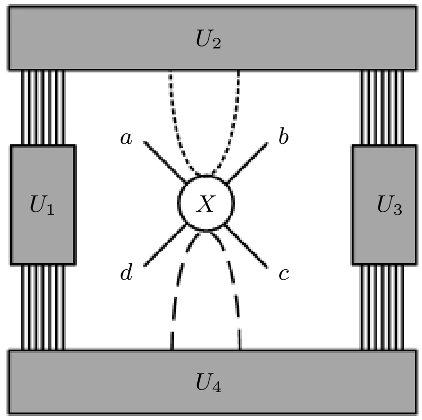


**Fig. 4.** How the ordering of L-gadgets for a V-gadget representing variable $X$ can be obtained from the rotation system of $X$.

adjacent to $l^-$ and a membrane edge, the membrane has to be drawn such that it encloses $l^+$. However, this is not possible, since every $l^+$ vertex of an L-gadget has at least four incident edges. □

Similar to the variable vertices in C-gadgets, a literal vertex lies *inside* if its "In" edge is crossed by a membrane edge, which implies that its clause edge (if existent) is crossed by another membrane edge ($l^+$ in Fig. 5(c)). A literal vertex lies *outside* if all its anchor edges are crossed by membrane edges ($l^+$ in Fig. 5(a)).

Let $X$ be a variable of a planar 3-SAT expression and $v \in V$ be the vertex in $G$ that corresponds to $X$. We construct the V-gadget of $X$ as follows (see Fig. 6(a) for the result). The V-gadget is attached to a U-graph $U$ that is adjacent to the face corresponding to $v$ in $G_S^*$. First attach a terminal gadget $t_1$ to $U$. Then, subsequently attach a positive or negative L-gadget depending on the occurrences of $X$ according to a total order obtained from the rotation system of $v$ such that ropes which are attached to the V-gadget do not cross. Fig. 4 shows an example of how the ordering of L-gadgets in the V-gadget for variable $X$ is obtained from the rotation system of the vertex representing $X$. Suppose the V-gadget is attached to $U_2$ (shown by the dotted semi-ellipses in Fig. 4), then the ordering of the L-gadgets from left to right is $a, d, c, d$. If it is attached to $U_4$ (shown by the dashed semi-ellipses in Fig. 4), then the ordering is $d, a, b, c$.

Intuitively, the planarity of $G$ is preserved in $G_S^*$. Append the second terminal $t_2$. For each L- and terminal gadget, connect its "In" edge with the "Out" edge of its immediate neighbor where "In" of $t_1$ is connected to "Out" of $t_2$ by a path, called *outer membrane*. The number of edges of the outer membrane is the number of occurrences of $X$ plus 1.

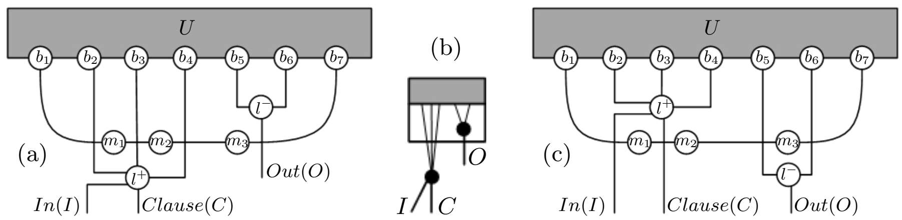


**Fig. 5.** 1-planar embedding of a positive literal gadget if the variable is true (a) or false (c), respectively. (b) Abbreviation for the "true state" of a positive literal gadget.

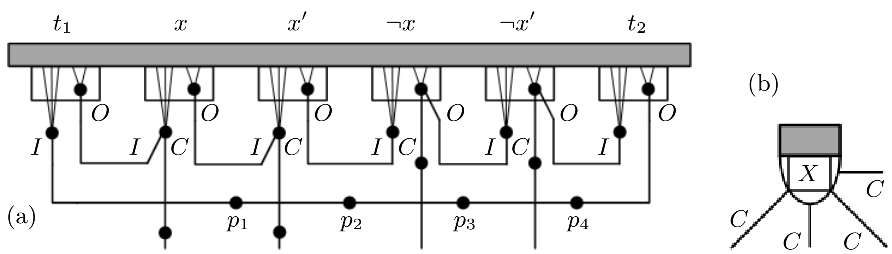


**Fig. 6.** (a) A variable gadget consisting of two positive, two negative and two terminal gadgets. The value of the variable is true. (b) Its abbreviation.

The last part of our reduction are the barriers and ropes. By the *size* of a barrier or rope we refer to the number of their edges. Korzhik and Mohar [11] proved that barriers of size at least 7 result in unique 1-planar embedded U-supergraphs. Let $l$ be the maximum number of occurrences of a variable in the given SAT expression. For the size of the barriers, we choose $\max\{7, l+2\}$. Note that the size of the outer membrane of a V-gadget is the number of times the corresponding variable occurs in the expression plus 1. Consequently, an outer membrane has strictly fewer edges than a barrier. Thus, an outer membrane can never cross a barrier. For every edge of $G$, a rope connecting the V-gadget with the C-gadget is introduced in $G^*_S$. More exactly, a rope always connects one of the literal vertices of an L-gadget with one of the variable vertices of a C-gadget such that the planar rotation system of $G$ is respected. The size of a rope is the size of a barrier plus 2. Figure 7 shows an example for a rope $r = \{l^+, r_1\}, \{r_1, r_2\}, \ldots, \{r_8, v\}$ with size 9. A rope crosses each edge of a barrier exactly once. Then, the rope crosses either two edges of a V-gadget (Fig. 7(b)) or one edge of a V-gadget and one edge of a C-gadget (Fig. 7(a)). Crossings of the first and last edge of a rope (e. g., $\{l^+, r_1\}, \{r_8, v\}$ of $r$) propagate the truth assignment of the literal at the one end to the clause at the other end. Consider again Fig. 7(a), where the positive L-gadget $x$, belonging to the V-gadget of variable $X$, is connected to C-gadget $C$. In the figure, $X$ is assigned true as $l^+$ lies outside. The rope propagates the truth assignment to the clause, where variable vertex $x$ can then lie inside and, hence, the clause is satisfied by $X$. Consider now Fig. 7(b), where the variable $X$ is assigned false and, hence, $l^+$ lies inside. Consequently, edge $\{l^+, r_1\}$ is crossed by the membrane of $x$ and edge $\{r_1, r_2\}$ is crossed by the outer membrane ($\{p, p'\}$). Hence, every remaining edge of $r$ is crossed by the barrier and, therefore, the variable vertex of $C$ must lie outside, representing that the clause is not satisfied by $x$. Now suppose that every literal of the clause is assigned false. Thus, all three variable vertices of $C$ lie outside. However, then there is no 1-planar drawing of $C$ and, hence, no 1-planar drawing of $G^*_S$. Hence, $C$ and, thus, the whole 3-SAT expression is not satisfiable.

We are now ready to prove Lemma 1:

*Proof.* Denote by $f$ a triangular face adjacent to a boundary edge of $G^*_S$. The lemma holds for the U-supergraph, i. e., $G^*_S$ without C- and V-gadgets, and ropes as the U-supergraph has a unique 1-planar embedding. Let $f$ be a triangular face adjacent to a boundary edge. Due to the unique embedding, no U-graph of the U-supergraph can be drawn inside of $f$ (cf. Fig. 1(a)). The same also holds true for every vertex $v$ of a C- and V-gadget, or a rope as in each case $v$ has at least degree two. If $v$ lies inside of $f$, it would cause at least two crossings of a boundary edge. Consequently, a boundary edge can only be crossed if a whole C- or V-gadget, or rope lies inside $f$. In the case of C- and V-gadgets, this is impossible even for the membrane or outer membrane of V-gadgets as the rotation system forces the first and last edges of the membrane to leave one of its endpoints outside of $f$. These two edges alone would already cause two crossings of the boundary edge. Similarly, as a rope connects a V-gadget with a C-gadget, it cannot be drawn inside of $f$. □

Before we can prove the main theorem, we need two additional lemmata.

**Lemma 4.** *Let $x$ be an L-gadget of a V-gadget $X$. Then, in every 1-planar drawing of $G^*_S$ respecting the given rotation system, the rope attached to $x$ is crossed by the outer membrane of $X$.*

*Proof.* In order to avoid a crossing of the rope, the outer membrane of $X$ has to be drawn "around" the C-gadget that is connected to $x$, i. e., the outer membrane encloses the C-gadget. However, then the outer membrane needs to cross at least one barrier, which is impossible since the size of the outer membrane is less than the size of a barrier. □

**Lemma 5.** *Let $X$ be a V-gadget. In every 1-planar drawing of $G^*_S$ respecting the given rotation system, all positive literal vertices $l^+$ of $X$'s L-gadgets lie inside if and only if all negative literal vertices $l^-$ of $X$'s L-gadgets lie outside.*

*Proof.* Let $x_1$ and $x_2$ be any positive or negative L-gadget part of $X$ and $l_1^+, l_1^-$ ($l_2^+, l_2^-$) be the literal vertices of $x_1$ ($x_2$). By Lemma 3, each of these literal vertices lie either inside or outside. It is not possible that both a positive and a negative literal vertex lie outside since a membrane of an L-gadget has size 4, whereas the literal vertices have a total of 5 anchor edges. Now suppose for

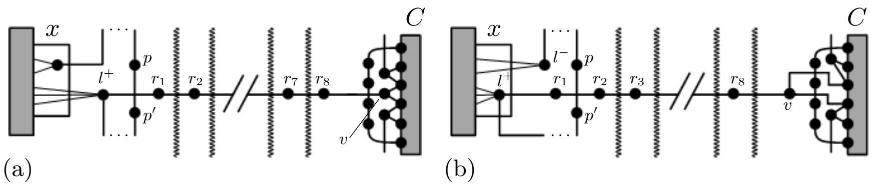


**Fig. 7.** From left to right: Fragment of a variable gadget that shows a literal gadget and a part of its outer membrane; A barrier drawn as curled lines; A clause gadget. (a) The literal $x$ is true, hence the clause represented by the C-gadget on the right is satisfied by $v$. (b) The literal $x$ is false, hence the clause is not satisfied by $v$.

contradiction that both $l_1^+$ and $l_2^-$ lie outside. As $l_1^+$ lies outside, $l_1^-$ lies inside and, consequently, the "Out" edge of $l_1^-$ is crossed by the membrane of $x_1$. Let $l_3^+$ be the positive literal vertex connected to $l_1^-$ via its "Out" edge. Vertex $l_3^+$ is "tugged" outside, i. e., $l_3^+$ cannot lie inside as its "In" edge (which is the same edge as the "Out" edge of $l_1^-$) is already crossed. If $l_3^+ = l_2^+$, then $l_2^+$ lies outside and, hence, $l_2^-$ must lie inside; a contradiction. Otherwise, $l_3^+$ belongs either to a terminal gadget or to another L-gadget. If $l_3^+$ belongs to a terminal gadget $t$, then the negative literal vertex $l_3^-$ of $t$ must lie inside. Via the outer membrane, the information that $l_3^-$ lies inside is propagated to the other terminal gadget $t'$ by the same mechanism that governs the ropes. Hence, the negative literal vertex of $t'$ lies inside and the positive one lies outside. If $l_3^+$ belongs to another L-gadget, then also the negative literal vertex lies inside and the positive one lies outside. By subsequently applying these arguments, we eventually arrive at $x_2$ and can conclude that $l_2^-$ must also lie inside; a contradiction. The reasoning if $l_1^-$ and $l_2^+$ lie outside is similar, as is the reasoning if both lie inside. □

**Theorem 1.** *1-planarity is* **NP***-hard for a graph with a given rotation system.*

*Proof.* A planar 3-SAT expression $\alpha$ is satisfiable if and only if the graph $G_S^*$ obtained from $\alpha$ is 1-planar.
"⇒": Draw the V-gadgets according to a satisfying truth assignment of the variables, i. e., the positive literal vertices of a variable gadget lie outside if and only if the corresponding variable is assigned true. Then, every C-gadget has a variable vertex that can lie inside and, thus, has a 1-planar drawing due to (A) and (B).
"⇐": We obtain a truth assignment of the variables from a 1-planar drawing of $G_S^*$ as follows. A variable is assigned true if and only if the positive literal vertices of the corresponding V-gadget lie outside. The so obtained assignment is consistent by Lemma 5. In each C-gadget, at least one variable vertex lies inside. This vertex is connected, via a rope, to a literal vertex of a V-gadget which necessarily lies outside. Thus, the corresponding variable satisfies the clause at hand. Hence, the obtained truth assignment satisfies the 3-SAT expression. □

In contrast, 1-planarity is solvable in linear time for embedded graphs. Given an embedding of a graph $G$ we first check whether an edge occurs in more than two faces. Then, we compute the planarization $G^\times$ of $G$ and check its planarity.

**Theorem 2.** *1-planarity can be solved in linear time for a graph with a given embedding.*

When we developed the **NP** reduction, we already had in mind that the reduction also works without a fixed rotation system. The terminal gadgets have the sole purpose that the literal vertices are inside the outer membrane. Otherwise, their "In" and "Clause" edges could avoid a crossing with the outer membrane by swapping their positions in the rotation system of the literal vertex. Also note that the given rotation system is optimal in the sense that a 1-plane drawing of $G^*_S$ would imply the defined rotation system, at least at the crucial parts, i. e., the gadgets. Our construction for the **NP**-hardness proof also holds if the rotation system is ignored, and we have an alternative proof to [11].

**Corollary 1.** *1-planarity is* **NP***-hard*

Next we address the crossing number problem, which asks whether there is a drawing of a graph in the plane with at most $k$ edge crossings. The **NP**-hardness of this problem was first proved by Garey and Johnson [7] using graphs with parallel edges and vertices of very high degrees. This problem has been addressed from various sides since then. Hliněný [9] improved the **NP**-hardness result to simple cubic graphs.

Here, we add another improvement and show that the crossing number remains **NP**-hard even if each edge is crossed at most once and the rotation system is given. To this effect we must only modify our V- gadgets.

**Theorem 3.** *Crossing number is* **NP***-hard for 1-planar graphs, even with a given rotation system.*

*Proof.* We reduce from the **NP**-complete planar vertex cover problem [6]. Its input is a planar graph $G = (V, E)$ and a non-negative integer $k$ and it asks whether there is a subset $V' \subseteq V$ with $|V'| \leq k$ such that every edge of $G$ is incident to at least one vertex in $V'$. $V'$ is then called a *vertex cover* of $G$.

Fix any planar embedding of $G$. As in Sect. 3, we start by transforming $G$ into a $U$-supergraph $G^*_S$. Again, each pair of U-graphs corresponding to adjacent vertices of $G^*$ is connected by $l = 7$ *barrier* edges. In the following we describe the single type of gadget which is attached to the U-graphs of $G^*_S$ for every vertex of $G$ in the same way as C- and V-gadgets were attached in the reduction of Sect. 3. Let $v \in V$, and $d$ be the degree of $v$. See Fig. 8 for an example of the gadget in the case of $d = 3$. The gadget for $v$ is attached to boundary vertices $b_1, \ldots, b_{d+3}$ of a U-graph and contains the *membrane vertices* $m_1, \ldots, m_d$ and the *connector* vertex $c_v$. There is a path called the *membrane* going from $b_1$ via $m_1, \ldots, m_d$ to $b_{d+2}$. We introduce $d + 1$ *anchor* edges connecting $c_v$ to the vertices $b_2, \ldots, b_{d+2}$. For every edge $\{u, v\} \in E$ we add a path from $c_u$ to $c_v$ consisting of exactly $l + 1$ edges, called the *rope*. We can choose the rotation

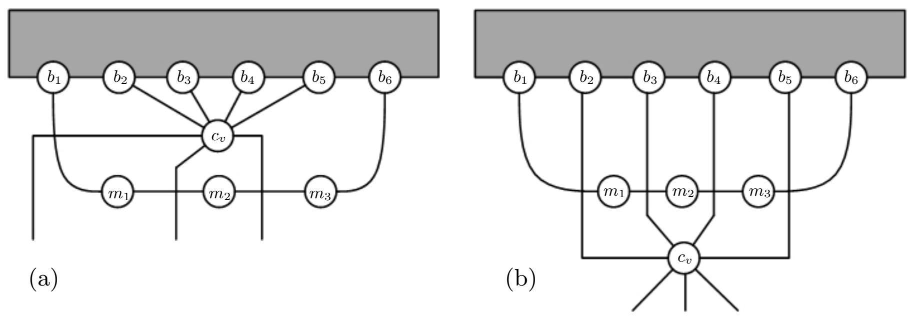


**Fig. 8.** The gadget used instead of variable gadgets for proving that crossing number is **NP**-hard for 1-planar graphs.

system at $c_v$ according to the embedding of $G$ such that the ropes do not cross. In general, a gadget has two possible 1-planar embeddings, where $c_v$ is either placed *inside*, i. e., the membrane is crossed by the $d$ ropes, or *outside*, i. e., the membrane is crossed by the $d+1$ anchor edges. Note that the latter case yields one more crossing. Given a 1-planar embedding of $G_S^*$, we define the set $V' \subseteq V$ by containing exactly those vertices $v$, whose corresponding connector vertices $c_v$ are placed outside. For every edge $\{u, v\} \in E$, the rope connecting $c_u$ with $c_v$ has to cross $l$ barrier edges. As it consists of only $l+1$ edges, it has only one edge left for crossing a membrane. Thus, at least one of the vertices $c_u$ or $c_v$ must be placed outside and $V'$ is a vertex cover of $G$. As the U-graphs have a unique 1-planar embedding, let $C$ be the constant number of crossings they contain. The total number of crossings of all membranes is $\sum_{v \in V} \deg(v) = 2m$ plus the number of connector vertices placed outside. Additionally, each of the $l \cdot m$ barrier edges are crossed by a rope. Now let $k' = C + (2+l)m + k$.

If there is a 1-planar embedding of $G_S^*$ with at most $k'$ crossings, at most $k$ connector vertices can be placed outside, i. e., $V'$ is a vertex cover with $|V'| \leq k$. Conversely suppose there is a vertex cover $V'$ with $|V'| \leq k$. Specify the 1-planar embedding by placing exactly those connector vertices $c_v$ outside with $v \in V'$. Then the resulting embedding of $G_S^*$ has exactly $C + (2+l)m + k$ crossings. □

## 4 Maximal 1-Planarity Testing

As 1-planarity is **NP**-complete, we are interested in cases with efficient solutions. Where is the room between an embedding and a rotation system? Maximality bridges the gap.

**Theorem 4.** *There is an efficient algorithm to test whether a graph with a given rotation system is maximal 1-planar.*


**Fig. 9.** The (a) planar and (b) non-planar drawing of the $K_4$.

*Proof.* Suppose the edges $(a, c)$ and $(b, d)$ cross. Then $G$, being maximal, contains the edges $(a, b), (b, c), (c, d), (d, a)$. Hence, there is a $K_4$ implied by the crossing (see also [19]). Conversely, consider a $K_4$ with the vertices $a, b, c, d$ as a subgraph of a 1-planar graph. There are two drawings: planar and non-planar; see Figs. 9(a) and 9(b). These drawings are distinguished by the rotation system, which is $a : (dcb), b : (acd), c : (adb), d : (abc)$ in the planar and $a : (cdb), b : (acd), c : (adb), d : (acb)$ in the non-planar case. Replace every non-planar $K_4$ by a planar subgraph with a crossing point at the intersection of the crossing edges, and run an adapted planarity test which tests if the given rotation system is planar.

Since there are at most $4n - 8$ edges, all $K_4$s can be searched in quadratic time by testing all pairs of disjoint edges if the remaining four edges are present. After the planarized embedding has been obtained, we can check for each pair of non-adjacent vertices if they can be joined by a new edge. Two vertices can be joined, if they are adjacent to a common face or if they are adjacent to faces with a common uncrossed edge. □

## 5 Conclusion and Perspectives

1-planar graphs are not well explored, in particular, in relation to planar graphs. We have added some tractable and some intractable instances. For instance, the classification of 1-planarity in parameterized complexity is open. Here the canonical parameter is the number of pairs of crossing edges. Also, we would like to find more instances where 1-planarity is efficiently solvable. Finally, we are interested in 'nice' drawings of 1-planar graphs.

## References

1. Battista, G.D., Eades, P., Tamassia, R., Tollis, I.G.: Graph Drawing: Algorithms for the Visualization of Graphs. Prentice-Hall (1999)
2. Bertolazzi, P., Battista, G.D., Liotta, G., Mannino, C.: Upward Drawings of Triconnected Digraphs. Algorithmica 12(6), 476–497 (1994)
3. Brandenburg, F.J., Eppstein, D., Gleißner, A., Goodrich, M.T., Hanauer, K., Reislhuber, J.: On the density of maximal 1-planar graphs (Jun 2012), under review
4. Eades, P., Hong, S.H., Liotta, G., Poon, S.H.: Fáry's Theorem for 1-Planar Graphs. In: Proceedings of the 18th Annual International Computing and Combinatorics Conference (COCOON) (2012), to appear

5. Fabrici, I., Madaras, T.: The Structure of 1-Planar Graphs. Discrete Mathematics 307(7–8), 854 – 865 (2007)
6. Garey, M.R., Johnson, D.S.: The Rectilinear Steiner Tree Problem is NP-Complete . SIAM J. Appl. Math. 32(4), 826–834 (1977)
7. Garey, M.R., Johnson, D.S.: Crossing number is np-complete. SIAM J. Matrix Anal. A. 4(3), 312–316 (1983)
8. Garg, A., Tamassia, R.: On the Computational Complexity of Upward and Rectilinear Planarity Testing. SIAM J. Comput. 31(2), 601–625 (2002)
9. Hliněný, P.: Crossing number is hard for cubic graphs. J. Comb. Theory Ser. B 96(4), 455–471 (Jul 2006)
10. Kaufmann, M., Wagner, D.: Drawing Graphs, LNCS, vol. 2025. Springer (2001)
11. Korzhik, V., Mohar, B.: Minimal Obstructions for 1-Immersions and Hardness of 1-Planarity Testing. In: Tollis, I., Patrignani, M. (eds.) Graph Drawing. LNCS, vol. 5417, pp. 302–312. Springer Berlin / Heidelberg (2009)
12. Korzhik, V.P.: Minimal Non-1-Planar Graphs. Discrete Math. 308(7), 1319–1327 (2008)
13. Lichtenstein, D.: Planar Formulae and their Uses. SIAM J. Comput. 11(2), 329–343 (1982)
14. Mohar, B.: Graphs on surfaces and their applications. Combinatorics, Probability & Computing 15(6), 939–941 (Nov 2006)
15. Nishizeki, T., Rahman, S.: Planar Graph Drawing. World Scientific (2004)
16. Pach, J., Tóth, G.: Graphs Drawn with Few Crossings per Edge. Combinatorica 17, 427–439 (1997)
17. Pelsmajer, M.J., Schaefer, M., Štefankovič, D.: Crossing Number of Graphs with Rotation Systems. Algorithmica 30, 679–702 (2011)
18. Ringel, G.: Ein Sechsfarbenproblem auf der Kugel. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 29, 107–117 (1965)
19. Suzuki, Y.: Re-embeddings of Maximum 1-Planar Graphs. SIAM J. Discrete Math. 24(4), 1527–1540 (2010)