A Note on Escape Rates

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1 Preliminaries

If \((X,d)\) is a compact metric space, we let \(C(X)\) denote the set of all continuous functions from \(X\) to \(\mathbb{R}\), which is a Banach space equipped with the supremum norm, \(\|f\| := \sup_{x\in X} |f(x)|\). Moreover, given a continuous map \(f: X \to X\) and \(\varphi \in C(X)\), we define for every \(n \in \mathbb{N}\) and \(x \in X\)

\[
S_n \varphi(x) := \sum_{i=0}^{n-1} \varphi(f^i(x)).
\]

If \(\mu\) is a Borel probability measure on \(X\), we define the measure \(f^* \mu\) by

\[
(f^* \mu)(A) := \mu(f^{-1}(A)) \quad \text{for all Borel sets } A \subset X.
\]

Then \(f^* \mu\) is also a Borel probability measure. The measure \(\mu\) is called \(f\)-invariant if \(f^* \mu = \mu\).

By \(\mathcal{M}\) we denote the set of all Borel probability measures on \(X\), and by \(\mathcal{M}(f)\) the set of all \(f\)-invariant elements of \(\mathcal{M}\), i.e., the fixed point set of the function \(f^*: \mathcal{M} \to \mathcal{M}\).

The following proposition summarizes some facts about (invariant) measures which will be used later.

1.1 Proposition:

1. A measure \(\mu \in \mathcal{M}\) is \(f\)-invariant iff \(\int \varphi d\mu = \int \varphi \circ f d\mu\) for all \(f \in L^1(X, \mu)\).

2. The set \(\mathcal{M}\) is compact and metrizable with respect to the weak* topology\(^1\), i.e., the topology defined by

\[
\mu_n \to \mu \quad \Leftrightarrow \quad \int \varphi d\mu_n \to \int \varphi d\mu \quad \text{for all } \varphi \in C(X).
\]

Moreover, the function \(f^*: \mathcal{M} \to \mathcal{M}\), induced by a continuous function \(f: X \to X\), is continuous.

\(^1\)According to the Riesz theorem, the set \(C(X)^*\) (the continuous dual of \(C(X)\)) can be identified with the space of all Radon measures on \(X\). Thus, the space of these measures can be equipped with the weak*-topology. By the theorem of Banach-Alaoglu, the closed unit ball in \(C(X)^*\) is weak*-compact, hence also its subset of functionals \(L\) with \(L(1) = 1\) and \(L(f) \geq 0\) for \(f \geq 0\) (the one corresponding to the probability measures).
3. The set $\mathcal{M}(f)$ is nonempty. (This is the theorem of Krylov-Bogolyubov.)

1.2 Lemma: Let $\mu \in \mathcal{M}$. Then, for each $\delta > 0$, there exists a finite measurable partition $\xi = \{C_1, \ldots, C_k\}$ of $X$ with $\text{diam}(C_i) < \delta$ for $i = 1, \ldots, k$ and $\mu(\partial\xi) = 0$, where $\partial\xi := \bigcup_{i=1}^{k} \partial C_i$.

**Proof:** For each $x \in X$, let us consider the disjoint uncountable union $\bigcup_{\varepsilon \in (0, \delta)} \partial B_{\varepsilon}(x)$, which has finite measure. Let us assume to the contrary that $\mu(\partial B_{\varepsilon}(x))$ is positive for every $\varepsilon \in (0, \delta)$. Then $(0, \delta)$ is the (countable) union of the sets $I_n := \{\varepsilon \in (0, \delta) : \mu(\partial B_{\varepsilon}(x)) > 1/n\}$, $n \in \mathbb{N}$. Hence, one of these sets $\mu(\partial B_{\varepsilon}(x))$ is positive for every $\varepsilon \in (0, \delta)$.

1.3 Lemma: If $\sum_{i=1}^{n} p_i = 1$, $p_i \geq 0$, $a_i \in \mathbb{R}$ and $A := \sum_{i=1}^{n} e^{a_i}$, then $\sum_{i=1}^{n} p_i (a_i - \log p_i) \leq \log A$ with equality if $p_i = e^{a_i}/A$.

**Proof:** Set $\alpha_i := e^{a_i}/A$, $x_i := p_i/\alpha_i$. Since the function $x \mapsto x \log(x)$ is strictly convex and $\sum \alpha_i = 1$, we have

$$0 = 1 \cdot \log(1) \leq \sum_{i=1}^{n} \alpha_i x_i \log(x_i) = \sum_{i=1}^{n} p_i \left(\log(p_i) + \log(A) - a_i\right)$$

with equality if and only if $p_i = e^{a_i}/A$. Thus, the metric entropy of $f$ with respect to $\mu$ is defined by $H_{\mu}(\xi) := -\sum_{C \in \xi} \mu(C) \log \mu(C)$ with the convention $0 \cdot \log(0) = 0$. Let $\mu \in \mathcal{M}(f)$. Then, for any countable measurable partition $\xi$ of $X$ we define $h_{\mu}(f, \xi) := \lim_{n \to \infty} \frac{1}{n} H_{\mu}(\xi_{-n})$.

The metric entropy of $f$ with respect to $\mu$ is defined by

$$h_{\mu}(f) := \sup_{\xi} h_{\mu}(f, \xi),$$

and the pressure of $f$ with respect to $\mu$ and $\varphi \in C(X)$ is defined by

$$P_{\mu}(f, \varphi) := h_{\mu}(f) + \int \varphi d\mu.$$
1.4 Proposition:

1. For an $f$-invariant measure $\mu$ and measurable partitions $\xi$ and $\eta$ it holds that
$$H_\mu(\xi \vee \eta) \leq H_\mu(\xi) + H_\mu(\eta).$$

2. For two $f$-invariant measures $\mu$ and $\lambda$, $p \in [0, 1]$, and a measurable partition $\xi$, we have
$$pH_\mu(\xi) + (1-p)H_\lambda(\xi) \leq pH_p\mu + (1-p)\lambda(\xi).$$

For each $n \in \mathbb{N}$, the Bowen-metric on $X$ is defined as
$$d_{n,f}(x, y) := \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)).$$

This metric is topologically equivalent to the given metric $d$. The ball of radius $\varepsilon > 0$ around $x \in X$ with respect to $d_{n,f}$ is denoted by $B_\varepsilon(x, n)$. These balls are also called Bowen-balls. A subset $E \subset X$ is called $(n, \varepsilon)$-separated if $d_{n,f}(x, y) \geq \varepsilon$ holds for any two distinct elements $x, y \in E$. A set $F$ $(n, \varepsilon)$-spans another set $G$ if for each $x \in G$ there is $y \in F$ with $d_{n,f}(x, y) < \varepsilon$, or equivalently, the Bowen-balls $B_\varepsilon(x, n), x \in F$, form an open cover of $G$.

2 Pressure Lemma

In this section, we give a detailed proof of Katok & Hasselblatt [5, Lem. 20.2.3], which is used in the proof of the variational principle for pressure.

2.1 Lemma: Let $(X, d)$ be a compact metric space, $f : X \to X$ a homeomorphism, and $\varphi \in C(X)$. Fix $\varepsilon > 0$ and select for each $n \in \mathbb{N}$ a $(n, \varepsilon)$-separated set $E_n \subset X$. Define sequences of Borel probability measures on $X$ by
$$\nu_n := \left( \sum_{x \in E_n} e^{S_n \varphi(x)} \right)^{-1} \sum_{x \in E_n} e^{S_n \varphi(x)} \delta_x$$

and
$$\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} f^i \nu_n.$$ 

Then there exists a limit point $\mu$ of $(\mu_n)_{n \in \mathbb{N}}$ (in the weak$^*$-topology) and any such $\mu$ is $f$-invariant and satisfies
$$\limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in E_n} e^{S_n \varphi(x)} \leq P_\mu(f, \varphi).$$
Proof: The proof is subdivided into three steps.

Step 1. Existence of a limit point µ follows from weak*-compactness (and metrizability). The $f$-invariance of $\mu$ is proved as follows: Let $\mu_{m_n} \to \mu$ in the weak*-topology. Then, since $f_*$ is continuous with respect to the weak*-topology, we have

$$f_*\mu = f_* \lim_{n \to \infty} \mu_{m_n} = \lim_{n \to \infty} f_* \mu_{m_n} = \lim_{n \to \infty} \frac{1}{m_n} \sum_{i=0}^{m_n-1} f_*^{i+1} \nu_{m_n}$$

$$= \lim_{n \to \infty} \frac{1}{m_n} \left[ \sum_{i=0}^{m_n-1} f_*^{i+1} \nu_{m_n} - \sum_{i=0}^{m_n-1} f_*^i \nu_{m_n} + \sum_{i=0}^{m_n-1} f_*^i \nu_{m_n} \right]$$

$$= \lim_{n \to \infty} \left( \frac{1}{m_n} [f_*^m \nu_{m_n} - \nu_{m_n}] + \mu_{m_n} \right).$$

Hence, we have to show that the weak*-limit of $\left( \frac{1}{m_n} [f_*^m \nu_{m_n} - \nu_{m_n}] + \mu_{m_n} \right)$ equals $\mu$. This holds, since for any $\psi \in C(X)$ we obtain

$$\left| \int \psi \left( \frac{1}{m_n} [f_*^m \nu_{m_n} - \nu_{m_n}] + \mu_{m_n} \right) \right| = \left| \int \psi \left( \frac{1}{m_n} [f_*^m \nu_{m_n} - \nu_{m_n}] \right) \right| = \left| \int \psi \left( \frac{1}{m_n} \int \nu_{m_n} - \int \psi \right) \right|$$

$$= \left| \frac{1}{m_n} \int \psi \circ f_*^m d\nu_{m_n} - \frac{1}{m_n} \int \psi d\nu_{m_n} + \int \psi d\mu_{m_n} - \int \psi d\mu \right|$$

$$\leq \frac{2}{m_n} \|\psi\| + \left| \int \psi d\mu_{m_n} - \int \psi d\mu \right| \to 0.$$

Step 2. By Lemma 1.2 there is a finite measurable partition $\xi$ of $X$ with diameter less than $\varepsilon$ and $\mu(\partial \xi) = 0$. It is easy to see that

$$n \int \varphi d\mu_n = \int S_n \varphi d\nu_n = \sum_{x \in E_n} S_n \varphi(x) \nu_n(\{x\}). \quad (1)$$

Since $E_n$ is $(n, \varepsilon)$-separated and diam $\xi < \varepsilon$, each element of the partition $\xi_{-n}$ contains at most one element of $E_n$, which implies

$$H_{\nu_n}(\xi_{-n}) = - \sum_{C \in \xi_{-n}} \nu_n(C) \log \nu_n(C) = - \sum_{x \in E_n} \nu_n(\{x\}) \log \nu_n(\{x\}). \quad (2)$$

Putting together (1) and (2), we get

$$H_{\nu_n}(\xi_{-n}) + n \int \varphi d\mu_n = \sum_{x \in E_n} \nu_n(\{x\}) \left( S_n \varphi(x) - \log \nu_n(\{x\}) \right). \quad (3)$$

If we write $E_n = \{x_1, \ldots, x_m\}$ and set $p_i = \nu_n(\{x_i\})$, $a_i := S_n \varphi(x_i)$, then Lemma 1.3 gives

$$H_{\nu_n}(\xi_{-n}) + n \int \varphi d\mu_n = \log \sum_{x \in E_n} e^{S_n \varphi(x)}.$$
Step 3. Suppose that $0 < q < n$. Then, obviously
\[
\frac{q}{n} \log \sum_{x \in E_n} e^{S_n \varphi(x)} \leq \frac{q}{n} \int \varphi d\mu_n + q \int \varphi d\mu_n.
\]

Let $a(k) := \lfloor (n - k)/q \rfloor$ be the integer part of $(n - k)/q$ whenever $0 \leq k < q$. Then $\{0, 1, \ldots, n-1\} = \{k + rq + i : 0 \leq r < a(k), \ 0 \leq i \leq q\} \cup S$, where $S = \{0, 1, \ldots, k-1, k + a(k)q, \ldots, n-1\}$, and $\#S \leq 2q$, since $k + a(k)q \geq n - q$ by definition of $a(k)$. Consequently,
\[
x_n = \left( \bigvee_{r=0}^{a(k)-1} f^{-(rq+k)}(\xi_{r,q}^f) \right) \lor \left( \bigvee_{i \in S} f^{-i}(\xi) \right)
\]
and, using Proposition 1.4 (1)
\[H_{\nu_n}(\xi_{r,q}^f) \leq \sum_{r=0}^{a(k)-1} H_{\nu_n}(f^{-(rq+k)}(\xi_{r,q}^f)) + \sum_{i \in S} H_{\nu_n}(f^{-i}(\xi)) \leq \sum_{r=0}^{a(k)-1} H_{f^{rq+k} \nu_n}(\xi_{r,q}^f) + 2q \log \#\xi.
\]

Putting this together with (4), we end up with
\[
\frac{q}{n} \log \sum_{x \in E_n} e^{S_n \varphi(x)} \leq \sum_{k=0}^{a(k)-1} \sum_{r=0}^{a(k)-1} H_{fn_k \nu_n}(\xi_{r,q}^f) + \frac{2q}{n} \log \#\xi + \frac{2q}{n} \log \#\xi + q \int \varphi d\mu_n,
\]
where the last inequality follows from Proposition 1.4 (2). Therefore,
\[
\limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in E_n} e^{S_n \varphi(x)} \leq \frac{1}{q} \lim_{n \to \infty} H_{\mu_n}(\xi_{r,q}^f) + \frac{1}{q} \int \varphi d\mu_n = \frac{1}{q} H_{\mu}(\xi_{r,q}^f) + \frac{1}{q} \int \varphi d\mu.
\]

The convergence $H_{\mu_n}(\xi_{r,q}^f) \to H_{\mu}(\xi_{r,q}^f)$ for $n \to \infty$ follows from the fact that $\mu(\partial \xi) = 0$ and hence, by $f$-invariance of $\mu$, also $\mu(\partial \xi_{r,q}^f) = 0$. Finally, letting $q \to \infty$, this gives the desired inequality.

\[\text{Here we also use the simple facts that } H_{\mu}(f^{-1}(\xi)) = H_{f \nu}(\xi) \text{ for every partition } \xi \text{ and every probability measure } \mu, \text{ and that } -\sum_{i=1}^n p_i \log(p_i) \leq \log(n) \text{ if } p_i \geq 0 \text{ and } \sum_i p_i = 1.
\]

\[\text{This follows from the Portmanteau-Theorem (cf. \[\text{[3]} \text{ Thm. 4.10]).}\]
3 Rates of Escape

Let $(X, d)$ be a compact metric space and $f : X \to X$ a homeomorphism. Moreover, let $m$ be a Borel measure on $X$ (called the reference measure). For any closed set $A \subset X$ we define the (upper) escape rate from $A$ by

$$\lambda := \lambda (f, m, A) := \limsup_{n \to \infty} \frac{1}{n} \log m(A_n), \quad A_n := \bigcap_{i=0}^{n-1} f^{-i}(A).$$

The following proposition is basically taken from Young [10, Thm. 1].

3.1 Proposition: Let $\varphi \in C(X)$ be a function such that there exist $C, \varepsilon > 0$ with

$$m(B_\varepsilon(x, n)) \leq C e^{-S_n \varphi(x)} \quad \text{for all } x \in X, \ n \in \mathbb{N}. \quad (5)$$

Assuming that $A_n \neq \emptyset$ for all $n \in \mathbb{N}$, there exists an $f$-invariant measure $\mu$ with $\text{supp} \mu \subset A$ and

$$\lambda \leq P_\mu (f, -\varphi) = h_\mu (f) - \int \varphi d\mu.$$

Proof: For each $n \in \mathbb{N}$, let $E_n$ be a maximal $(n, \varepsilon)$-separated set contained in $A_n$. Then $E_n$ also $(n, \varepsilon)$-spans the set $A_n$ (since otherwise there would exist $x \in A_n \setminus E_n$ such that $E_n \cup \{x\}$ is $(n, \varepsilon)$-separated) and hence

$$m(A_n) \leq m \left( \bigcup_{x \in E_n} B_\varepsilon(x, n) \right) \leq \sum_{x \in E_n} m(B_\varepsilon(x, n)) \leq C \sum_{x \in E_n} e^{-S_n \varphi(x)}.$$

Then Lemma 2.1 gives an $f$-invariant measure $\mu$ with

$$\lambda = \limsup_{n \to \infty} \frac{1}{n} \log m(A_n) \leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in E_n} e^{S_n (-\varphi(x))} \leq P_\mu (f, -\varphi),$$

as claimed. It remains to show that $\text{supp} \mu \subset A$. Recall that $\mu$ is the weak*-limit of a sequence $\mu_{m_n}$ with $\text{supp} \mu_{m_n} = \bigcup_{i=0}^{m_n-1} f^i(E_{m_n}) \subset A$. Assume to the contrary that $\mu(X \setminus A) > 0$. By regularity of Borel measures on separable complete metric spaces (theorem of Ulam), there exists a closed set $C \subset X \setminus A$ with $\mu(C) > 0$. Since metric spaces are normal, the lemma of Urysohn (cf. Kelley [6, Lem. 4]) yields a continuous function $\alpha : X \to [0, 1]$ with $\alpha(x) \equiv 0$ on $A$ and $\alpha(x) \equiv 1$ on $C$. Then $\int \alpha d\mu_{m_n} = 0$ for all $n$ and $\int \alpha d\mu_{m_n} \to \int \alpha d\mu \geq \int_C \alpha d\mu = \mu(C) > 0$, a contradiction.

One basic result which gives an estimate of type (5) for volumes of Bowen-balls is the Bowen-Ruelle volume lemma for basic sets of Axiom A systems. Different versions and variations of this result can be found in [1, 2, 4, 7, 8, 9, 10].
References


