A small gain condition for interconnections of ISS systems with mixed ISS characterizations

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Abstract—We consider interconnected nonlinear systems with external inputs. Each of the subsystems is assumed to be input-to-state stable (ISS). Sufficient conditions of small-gain type are provided guaranteeing that the interconnection is ISS. To this end we extend recently obtained small gain theorems to a more general type of interconnections. The small gain theorem proved here is applicable to situations where the ISS conditions are formulated differently for each subsystem and are either given in the maximization or the summation sense. An example shows the advantages of our results in comparison with the known ones.

Index Terms—Control systems, nonlinear systems, large-scale systems, stability criteria, Lyapunov methods.

I. INTRODUCTION

Stability of nonlinear systems with inputs can be described in different ways as for example in sense of dissipativity [19], passivity [17], [18], input-to-state stability (ISS) and others. In this paper we consider interconnections of such systems and assume that each of them is characterized in terms of ISS. This notion was introduced by E. Sontag in 1989, see [15]. However the ISS property can be defined in several equivalent ways. The main question of the paper is whether an interconnection of several ISS systems is stable. In particular we are interested in a possibly sharp stability condition for the case when the ISS characterization of single systems are different. Moreover we will provide a construction of an ISS Lyapunov function for interconnections of such systems.

Starting with pioneering works [11], [10] stability of interconnections of ISS systems has been studied by many authors, see for example [13], [1], [3], [9]. In particular it is known that cascades of ISS systems are ISS, however, a feedback interconnection of two ISS systems is in general unstable. Some conditions applied on the gains of both systems can assure that their feedback is ISS. The first result of the small gain type was proved in [11] for a feedback of two ISS systems. The Lyapunov version of this result is given in [10]. Here we would like to note the difference between the small gain conditions in these papers. One of them states in [10] that the composition of both gains should be less than identity. The second condition in [11] is similar but it involves the composition of both gains and further functions of the form \((\text{id} + \alpha_i)\). This difference is due to the use of different definitions of ISS in both papers. Both definitions are equivalent but the gains enter as a maximum in the first definition, and a sum of the gains is taken in the second one. The results of [11] and [10] were generalized for an interconnection of \(n \geq 2\) systems in [4], [6]. It was pointed out that the similar difference in the small gain conditions remains, i.e., if the gains of different inputs enters as a maximum of gains in the ISS definition or a sum of them is taken in the definition. Moreover, it was shown that the auxiliary functions \((\text{id} + \alpha_i)\) are essential in the summation case and cannot be omitted [4].

A more general definition of ISS for the case of many inputs was introduced in [14], [5] and [7]. For recent results on the small-gain conditions for a wider class of interconnections we refer to [12], [8].

In some applications it may happen that the gains of a part of systems of an interconnection are given in maximization terms while the gains of another part are given in a summation formulation. This motivates the question: do we need functions \((\text{id} + \alpha_i)\) and how many of them in the small gain condition to assure stability in this case? In this paper we consider this case and answer this question. Namely we consider \(n\) interconnected ISS systems, such that in the ISS definition of the first \(k \leq n\) systems the gains enter additively. For the rest of systems the definition with maximum is used. We will see that the small gain condition provided in this paper is less conservative then the one used for the general definition in [14]. Our result contains the known small gain conditions from [4] as a special case \(k = 0\) or \(k = n\), i.e., if only one type of ISS definition is used. An example in the end of the paper shows the advantages of our results in comparison with the known ones.

This paper is organized as follows. In section II we present necessary notation and definitions. Section III provides some auxiliary lemmas necessary. A new small gain condition assuring stability of the considered interconnection is proved in section IV. Section Section V provides a construction of an ISS Lyapunov function. To demonstrate novelty and advantages of the new small gain condition an example is given in section VI. Section VII contains conclusions of the paper.

II. PRELIMINARIES AND PROBLEM STATEMENT

In the following we denote \(\mathbb{R}_+ := [0, \infty)\). \(\mathbb{R}^n_+\) is the positive orthant \(\{x \in \mathbb{R}^n : x \geq 0\}\). \(x^T\) stands for the transpose of a vector \(x \in \mathbb{R}^n\). For \(x, y \in \mathbb{R}^n\), we use the standard partial order induced by the positive orthant. It is given by

\[
\begin{align*}
x \geq y & \iff x_i \geq y_i, \quad i = 1, \ldots, n, \\
x > y & \iff x_i > y_i, \quad i = 1, \ldots, n.
\end{align*}
\]
We write \( x \geq y \iff \exists i \in \{1, \ldots, n\} : x_i < y_i \). For a nonempty index set \( I \subset \{1, \ldots, n\} \) we denote by \(|I|\) the number of elements of \( I \). We write \( y_I \) for restriction \( y_{(i)} \) of vectors \( y \in \mathbb{R}^n \). Let \( P_I \) denote the projection of \( \mathbb{R}^n \) onto \( \mathbb{R}^{|I|} \) and \( R_I \) be the anti-projection \( \mathbb{R}^{|I|} \to \mathbb{R}^n \), defined by

\[
x \mapsto \sum_{k=1}^{|I|} x_k e_i_k,
\]

where \( \{e_k\}_{k=1}^{n} \) denotes the standard basis in \( \mathbb{R}^n \) and \( I = \{i_1, \ldots, i_{|I|}\} \), with \( i_k < i_{k+1}, k = 1, \ldots, |I| - 1 \).

For a function \( v : \mathbb{R}_+ \to \mathbb{R}^m \) we define its restriction to the interval \([s_1, s_2] \) by

\[
v_{[s_1, s_2]}(t) = \begin{cases} v(t), & \text{if } t \in [s_1, s_2], \\ 0, & \text{otherwise}. \end{cases}
\]

A function \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be of class \( \mathcal{K} \) if it is continuous, strictly increasing and \( \gamma(0) = 0 \). It is of class \( \mathcal{K}_{\infty} \) if, in addition, it is unbounded. Note that for any \( \alpha \in \mathcal{K}_{\infty} \) its inverse function \( \alpha^{-1} \) always exists and \( \alpha^{-1} \in \mathcal{K}_{\infty} \).

A function \( \beta : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be of class \( \mathcal{KL} \) if, for each fixed \( t \), the function \( \beta(\cdot, t) \) is of class \( \mathcal{K} \) and, for each fixed \( s \), the function \( \beta(s, \cdot) \) is non-increasing and tends to zero for \( t \to \infty \).

By \( \| \cdot \| \) we denote the operator norm in \( \mathbb{R}^n \), and let in particular \( \|x\|_{\max} = \max_i |x_i| \) be the maximum norm. The essential supremum norm of a measurable function \( \phi \) is denoted by \( \|\phi\|_{\infty} \). \( L_\infty \) is the set of measurable functions for which this norm is finite.

Consider the system

\[
\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,
\]

and assume that the system is forward complete. This means that for all initial values \( x(0) \in \mathbb{R}^n \) and all essentially bounded measurable inputs \( u \) solutions exist for all positive times. Assume also that for any initial value \( x(0) \) and input \( u \) the solution is unique.

The following notions of stability are used in the remainder of the paper.

**Definition 2.1:** The system (1) is input-to-state stable (ISS), if there exist functions \( \beta \) of class \( \mathcal{KL} \) and \( \gamma \) of class \( \mathcal{K} \), such that the inequality

\[
|v(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_\infty)
\]

holds for all initial states \( x(0) \in \mathbb{R}^n \) and inputs \( u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m) \), \( t \geq 0 \).

**Definition 2.2:** The system (1) is globally stable (GS), if there exist functions \( \sigma, \gamma \) of class \( \mathcal{K} \), such that the inequality

\[
|v(t)| \leq \sigma(|x(0)|) + \gamma(\|u\|_\infty)
\]

holds for all \( x(0) \in \mathbb{R}^n \), \( u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m) \), \( t \geq 0 \).

**Definition 2.3:** The system (1) has the asymptotic gain (AG) property, if there exists a function \( \gamma \) of class \( \mathcal{K} \), such that the inequality

\[
\limsup_{t \to \infty} |x(t)| \leq \gamma(\|u\|_\infty)
\]

holds for all \( x(0) \in \mathbb{R}^n \) and \( u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m) \).

**Remark 2.4:** Instead of sum of the terms on the right-hand side of (2) one can take the maximum of these terms:

\[
|v(t)| \leq \max\{\beta(|x(0)|, t), \gamma(\|u\|_\infty)\}.
\]

This leads to an equivalent definition of ISS. Note that functions \( \beta, \gamma \) in (5) are in general different from those in (2). A similar equivalent definition can be written for the GS systems.

**Remark 2.5:** In [16] it was shown that a system is ISS if and only if it is GS and has the AG property.

Consider \( n \) interconnected control systems given by

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, \ldots, x_n, u_1) \\
\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n, u_n)
\end{align*}
\]

where \( x_i \in \mathbb{R}^{N_i}, u_i \in \mathbb{R}^{m_i} \) and the functions \( f_i : \mathbb{R}^{\sum_{j=1}^{k} N_j + m_i} \to \mathbb{R}^{N_i} \) are continuous and for all \( r \in \mathbb{R} \) are locally Lipschitz continuous in \( x = (x_1^T, \ldots, x_n^T)^T \) uniformly in \( u_i \) for \( |u_i| \leq r \). This regularity condition for \( f_i \) guarantees the existence and uniqueness of solution for the \( i \)th subsystem for a given initial condition.

The interconnection (6) can be written as (1) with \( x = (x_1, \ldots, x_n)^T, u = (u_1, \ldots, u_n)^T \) and

\[
f(x, u) = \begin{pmatrix}
 f_1(x_1, \ldots, x_n, P_1(u)) \\
 \vdots \\
 f_n(x_1, \ldots, x_n, P_n(u))
\end{pmatrix}.
\]

If we consider individual subsystems, we treat the state \( x_j, j \neq i \) as an independent input for the \( i \)th subsystem.

Consider an index set \( I := \{1, \ldots, n\} \) consisted of two non-overlapping subsets \( I_{\Sigma}, I_{\max} \) such that \( I_{\max} = I \setminus I_{\Sigma} \).

Let subsystems of (6) be ISS, i.e., there exist functions \( \beta_i \) of class \( \mathcal{KL}, \gamma_{ij}, \gamma_i \in \mathcal{K}_{\infty} \cup \{0\} \) such that for all initial values \( x_i(0) \) and inputs \( u \in \mathbb{R}^m \) there exists a unique solution \( x_i(t) \) satisfying for all \( t \geq 0 \)

\[
|v_i(t)| \leq \beta_i(|x_i(0)|, t) + \sum_{j=1}^{n} \gamma_{ij}(|x_j(0)|) + \gamma_i(\|u\|_\infty)
\]

for \( i \in I_{\Sigma} \) and

\[
|v_i(t)| \leq \max\{\beta_i(|x_i(0)|, t), \gamma_i(\|u\|_\infty)\}
\]

for \( i \in I_{\max} \).

**Remark 2.6:** Note that without loss of generality we can assume that \( I_{\Sigma} = \{1, \ldots, k\} \) and \( I_{\max} = \{k+1, \ldots, n\} \) where \( k := |I_{\Sigma}| \). This can be always achieved by an appropriate renumbering of the subsystems in (6).

We say that the gains in (7) are of sum type and the gains in (8) are of max type.

Since ISS implies GS and AG property, there exist functions \( \sigma_i, \tilde{\gamma}_{ij}, \tilde{\gamma}_i \in \mathcal{K} \cup \{0\} \), such that for any initial value \( x_i(0) \) and input \( u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m) \) there exists a unique solution \( x_i(t) \) and

\[
|v_i(t)| \leq \sigma_i(|x_i(0)|, t) + \sum_{j=1}^{n} \tilde{\gamma}_{ij}(|x_j(0)|) + \tilde{\gamma}_i(\|u\|_\infty)
\]
for $i \in I_{\Sigma}$ and
\[ |x_i(t)| \leq \max_j \{ \sigma_j(|x_j(0)|), \gamma_j(\|x_{j+1}(t)\|_\infty), \gamma_i(\|u\|_\infty) \} \]
for $i \in I_{\text{max}}$ for all $t \geq 0$ and there exist functions $\gamma_{ij}, \gamma_i \in K \cup \{0\}$, such that for any initial value $x_i(0)$ and input $u \in L_{\infty}(\mathbb{R}_+, \mathbb{R}^n)$ there exists a unique solution $x_i(t)$ and
\[ \limsup_{t \to \infty} |x_i(t)| \leq \sum_{j=1}^n \gamma_{ij}(\|x_j(0)\|_\infty) + \gamma_i(\|u\|_\infty) \]
for $i \in I_{\Sigma}$ and
\[ \limsup_{t \to \infty} |x_i(t)| \leq \max_j \{ \gamma_{ij}(\|x_j(0)\|_\infty), \gamma_i(\|u\|_\infty) \} \]
for $i \in I_{\text{max}}$.

Let us collect the gains $\gamma_{ij}$ in a matrix $\Gamma = (\gamma_{ij})_{n \times n}$, denoting $\gamma_{ii} = 0, i = 1, \ldots, n$. The operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is defined by
\[ \Gamma(s) := \{ \Gamma_1(s), \ldots, \Gamma_n(s) \}^T \]
where functions $\Gamma_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ are given by $\Gamma_i(s) := \gamma_{i1}(s_1) + \cdots + \gamma_{in}(s_n)$ for $i \in I_{\Sigma}$ and $\Gamma_i(s) := \max \{ \gamma_{i1}(s_1), \ldots, \gamma_{in}(s_n) \}$ for $i \in I_{\text{max}}$ for $s \in \mathbb{R}^n_+$. In particular if $I_{\Sigma} = \{1, \ldots, k\}$ and $I_{\text{max}} = \{k+1, \ldots, n\}$ we have
\[ \Gamma(s) = \begin{pmatrix} \gamma_{12}(s_2) + \cdots + \gamma_{1n}(s_n) \\ \vdots \\ \gamma_{k1}(s_1) + \cdots + \gamma_{kn}(s_n) \\ \max \{ \gamma_{k+1,1}(s_1), \ldots, \gamma_{k+1,n}(s_n) \} \\ \vdots \\ \max \{ \gamma_{n1}(s_1), \ldots, \gamma_{nn-1}(s_{n-1}) \} \end{pmatrix} \]

Interconnections of such systems were considered in [4] for $I_{\Sigma} = \emptyset$ and $I_{\text{max}} = \emptyset$. In [14], [7] more general formulations of ISS are considered, which encompass the case studied in this paper. However, in these references the specific results available for the structure considered here are not provided.

Our main question is whether the interconnection (6) is ISS from $u$ to $x$. It is known that even if all subsystems are ISS their interconnection need not be ISS. Recall the small gain conditions for the cases $I_{\Sigma} = \emptyset$ and $I_{\text{max}} = \emptyset$ assuring ISS property of such interconnections from [4]:
\[ \Gamma \circ D(s) \not\supset s, \forall s \in \mathbb{R}^n_+ \setminus \{0\} \]
for some $D := \text{diag}(\text{id} + \alpha), \alpha \in K_\infty$ for $I_{\text{max}} = \emptyset$ and
\[ \Gamma(s) \not\supset s, \forall s \in \mathbb{R}^n_+ \setminus \{0\} \]
for $I_{\text{max}} = \emptyset$. In case both $I_{\Sigma}$ and $I_{\text{max}}$ are not empty we can use
\[ \max_{i=1, \ldots, n} \{ x_i \} \leq \sum_{i=1}^n x_i \leq n \max_{i=1, \ldots, n} \{ x_i \} \]
for $I_{\Sigma} = \emptyset$ or $I_{\text{max}} = \emptyset$. But this leads to more conservative gains. To avoid this conservativeness we are going to obtain a new small gain condition for $I_{\Sigma} \neq \emptyset \neq I_{\text{max}}$. Expressions in (15), (16) prompt us to consider the following small gain condition: Let the map $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be defined by
\[ D(s) := (D_1(s_1), \ldots, D_n(s_n))^T, \quad s \in \mathbb{R}^n_+ \]
where $D_i(s_i) := (\text{id} + \alpha)(s_i)$ for $i \in I_{\Sigma}$ and some $\alpha \in K_\infty$ and $D_i(s_i) := s_i$ for $i \in I_{\text{max}}$. The small-gain condition on the operator $\Gamma$ is then
\[ \Gamma \circ D(s) \not\supset s, \forall s \in \mathbb{R}^n_+ \setminus \{0\}. \]

In this paper we will prove that this small gain condition guarantees the ISS property of the interconnection (6) and show how an ISS-Lyapunov function can be constructed if this condition is satisfied.

III. AUXILIARY RESULTS

Before we proceed to the main results we prove some auxiliary results for the operators satisfying small gain condition (19).

Lemma 3.1: The small gain condition (19) is equivalent to $D \circ \Gamma(v) \not\supset v$ for all $v \in \mathbb{R}^n_+ \setminus \{0\}$.

Proof: Note that $D$ is always invertible and
\[ D^{-1}(v) := \text{diag}(D_1^{-1}(v_1), \ldots, D_n^{-1}(v_n)). \]

For every $v \in \mathbb{R}^n_+$ there exists a unique $w \in \mathbb{R}^n_+$ such that $v = D(w)$ and vice versa. By monotonicity of $D$ and $D^{-1}$ we have $D \circ \Gamma(v) \not\supset v$ if and only if $\Gamma(v) \not\supset D^{-1}(v)$. For any $w \in \mathbb{R}^n_+$ define $v := D(w)$. Then $\Gamma \circ D(w) \not\supset w$. This proves the equivalence.

For convenience let us introduce $\mu : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by
\[ \mu(w, v) := \{ \mu_1(w_1, v_1), \ldots, \mu_n(w_n, v_n) \}^T, \quad w, v \in \mathbb{R}^n_+, \]
where $\mu_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ is such that $\mu_i(w_i, v_i) := w_i + v_i$ for $i \in I_{\Sigma}$ and $\mu_i(w_i, v_i) := \max \{ w_i, v_i \}$ for $i \in I_{\text{max}}$. The following counterpart of Lemma 13 in [4] provides the main technical step in the proof of the main results.

Lemma 3.2: Let for some $D$ as in (18) the operator $\Gamma$ defined in (13) satisfy $\Gamma \circ D(s) \not\supset s$ for some $s \in \mathbb{R}^n_+ \setminus \{0\}$. Then there exists a $\phi \in K_\infty$ such that for all $w, v \in \mathbb{R}^n_+$,
\[ w \leq \mu(\Gamma(w), v) \]
implies $\|w\| \leq \phi(\|v\|)$.

Proof: Without loss of generality we assume $I_{\Sigma} = \{1, \ldots, k\}$ and $I_{\text{max}} = I \setminus I_{\Sigma}$, see Remark 2.6, and hence $\Gamma$ is as in (14). Fix any $v \in \mathbb{R}^n_+$. We first show, that for those $w \in \mathbb{R}^n_+$ satisfying (21) at least some components have to be bounded.

Let $\bar{D} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be defined by
\[ \bar{D}(s) := (s_1 + \alpha^{-1}(s_1), \ldots, s_k + \alpha^{-1}(s_k), s_{k+1}, \ldots, s_n)^T, \quad s \in \mathbb{R}^n_+ \]
and let
\[ s^* := \bar{D}(v). \]
Assume there exists \( w = (w_1, \ldots, w_n)^T \) satisfying (21) and such that \( w_i > s_i^*, \ i = 1, \ldots, n \). In particular, for \( i = 1, \ldots, k \) we have
\[
s_i^* < w_i \leq \gamma_{i1}(w_1) + \cdots + \gamma_{in}(w_n) + v_i
\] (22)
and hence from the definition of \( s^* \):
\[
s_i^* = v_i + \alpha^{-1}(v_i) < \gamma_{i1}(w_1) + \cdots + \gamma_{in}(w_n) + v_i.
\]
Then
\[
v_i < \alpha(\gamma_{i1}(w_1) + \cdots + \gamma_{in}(w_n)).
\]
From (22) it follows
\[
w_i \leq \gamma_{i1}(w_1) + \cdots + \gamma_{in}(w_n) + v_i < (id + \alpha) \circ (\gamma_{i1}(w_1) + \cdots + \gamma_{in}(w_n)).
\] (23)
Similarly, by the construction of \( w \) we have for \( i = k + 1, \ldots, n \)
\[
s_i^* < w_i \leq \max\{\gamma_{i1}(w_1), \ldots, \gamma_{in}(w_n), v_i\}.
\] (24)
From the definition of \( s^* \) we have
\[
s_i^* = v_i < w_i \leq \max\{\gamma_{i1}(w_1), \ldots, \gamma_{in}(w_n), v_i\}. \] (25)
Hence,
\[
w_i \leq \max\{\gamma_{i1}(w_1), \ldots, \gamma_{in}(w_n)\}.
\] (26)
From (23), (26) we get
\[
\begin{pmatrix}
(id + \alpha) \circ (\gamma_{12}(w_2) + \cdots + \gamma_{1n}(w_n)) \\
\vdots \\
(id + \alpha) \circ (\gamma_{k1}(w_1) + \cdots + \gamma_{kn}(w_n)) \\
\max\{\gamma_{k+11}(w_1), \ldots, \gamma_{k+n}(w_n)\} \\
\vdots \\
\max\{\gamma_{n1}(w_1), \ldots, \gamma_{nn-1}(w_{n-1})\}
\end{pmatrix},
\]
i.e., \( w \leq D \circ \Gamma(w) \). This contradicts the condition \( \Gamma \circ D(w) \not\geq w \) of our lemma which is equivalent to \( D \circ \Gamma(w) \not\leq w \) by Lemma 3.1. Hence some components of \( w \) are bounded. Iteratively we will prove that all components of \( w \) are bounded.

Let us denote \( s^1 := s^* \) from the first step. We have already proved that \( w \not\geq s^1 \) for all \( w \) satisfying (21). Fix such a \( w \), then there exists an index set \( I_1 \subset I \), possibly depending on \( w \), such that \( w_i > s^1_i, \ i \in I_1 \) and \( w_i \leq s^1_i, \ for \ i \in I \setminus I_1 \). Note that by the first step \( I_1 \) is nonempty. We now renumber the coordinates so that
\[
w_i > s^1_i \ and \ w_i \leq \sum_{j=1}^{n} \gamma_{ij}(w_j) + v_i, \ i = 1, \ldots, k_1, \] (27)
\[
w_i > s^1_i \ and \ w_i \leq \max_j \{\max_i \gamma_{ij}(w_j), v_i\}, \ i = k_1 + 1, \ldots, n_1, \] (28)
\[
w_i \leq s^1_i \ and \ w_i \leq \sum_{j=1}^{n} \gamma_{ij}(w_j) + v_i, \] (29)
i = n_1 + 1, \ldots, n_1 + 1 + k_2
\[
w_i \leq s^1_i \ and \ w_i \leq \max_j \{\max_i \gamma_{ij}(w_j), v_i\}, \] (30)
i = n_1 + k_2 + 1, \ldots, n_1 + k_2 + k. Using (29), (30) in (27), (28) we get:
\[
w_i \leq \sum_{j=1}^{n_1} \gamma_{ij}(w_j) + \sum_{j=n_1+1}^{n} \gamma_{ij}(w_j) + v_i, \] (31)
i = 1, \ldots, k_1, and
\[
w_i \leq \max_{j=1,\ldots, n_1} \gamma_{ij}(w_j), \max_{j=n_1+1,\ldots, n} \gamma_{ij}(w_j), v_i, \] (32)
i = k_1 + 1, \ldots, n_1. Define \( v^1 \) by
\[
v^1_i := \max_{j=n_1+1,\ldots, n} \gamma_{ij}(w_j), v_i, i = k_1 + 1, \ldots, n_1.
\]
Now (31), (32) take the form:
\[
w_i \leq \sum_{j=1}^{n_1} \gamma_{ij}(w_j) + v^1_i, \] (33)
i = 1, \ldots, k_1,
\[
w_i \leq \max_{j=1,\ldots, n_1} \gamma_{ij}(w_j), v^1_i, \] (34)
i = k_1 + 1, \ldots, n_1.
Let us represent \( \Gamma \) as \( \Gamma = \begin{pmatrix} \Gamma_{I_1,I_1} & \Gamma_{I_1,I_2} \\ \Gamma_{I_2,I_1} & \Gamma_{I_2,I_2} \end{pmatrix} \) and let us define the maps \( \Gamma_{I_1,I_1} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}, \Gamma_{I_1,I_2} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}, \Gamma_{I_2,I_1} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}, \Gamma_{I_2,I_2} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1} \) analogous to the \( \Gamma \). Let
\[
D_{I_1}(s) := (id + \alpha)(s), (id + \alpha)(s), (s_{k+1}, \ldots, s_n)^T.
\]
From \( \Gamma \circ D(s) \not\geq s \) for all \( s \not\geq 0, s \in \mathbb{R}^n \) it follows that \( \Gamma_{I_1,I_1} \circ D_{I_1}(z) \not\geq z \) for all \( z \not\geq 0, z \in \mathbb{R}^{n_1} \). Using the same approach as for \( w \in \mathbb{R}^n \) it can be proved that some components of \( w^1 = (w_1, \ldots, w_n)^T \) are bounded.
We proceed inductively, defining
\[
I_{j+1} := I \setminus I_j \] with \( I_{j+1} := I \setminus I_j \) and
\[
s^{j+1} := D_{I_j} \circ (\mu^j(s^*_{i_j}, v_{I_j})), \] (36)
where \( D_{I_j} \) is defined analogous to \( D \), the map \( \Gamma_{I_j,I_j} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1} \) acts analogous to \( \Gamma \) for vectors of the corresponding dimension, \( s^*_{I_j} = (s^*_{i_j})_{i \in I_j} \) is the restriction defined in the preliminaries and \( \mu^j \) is appropriately defined similar to the definition of \( \mu \).
The nesting (35), (36) will end after at most \( n-1 \) steps: there exists a maximal \( l \leq n \), such that
\[
I \supseteq I_1 \supseteq \cdots \supseteq I_l \not\supseteq 0
\]
and all components of \( w_{I_l} \) are bounded by the corresponding components of \( s^{l+1} \). Let
\[
s_\kappa := \max\{s^*, R_{I_1}(s^2), \ldots, R_{I_l}(s^{l+1})\}
\]
\[
\max\{(s^*)_1, (R_{I_1}(s^2))_1, \ldots, (R_{I_l}(s^{l+1}))_1\}
\]
\[
\vdots
\]
\[
\max\{(s^*)_n, (R_{I_1}(s^2))_n, \ldots, (R_{I_l}(s^{l+1}))_n\}
\]
where \( R_{l_j} \) denotes the anti-projection \( \mathbb{R}_{+l_j} \rightarrow \mathbb{R}_{+} \) defined above.

Let the n-fold composition \( M \circ \ldots \circ M \) be denoted by \([M]^n\).

By the definition of \( \mu \) for all \( v \in \mathbb{R}_{+}^n \) it holds
\[
0 \leq v \leq \mu(\Gamma_{\text{id}})(v) := \mu(\Gamma(v), v).
\]

Applying \( \tilde{D} \) we have
\[
0 \leq v \leq \tilde{D}(v) \leq \tilde{D} \circ \mu(\Gamma_{\text{id}})(v) \leq \ldots \leq [\tilde{D} \circ \mu(\Gamma_{\text{id}})]^n(v).
\]

(37)

From (36) and (37) for \( w \) satisfying (21) we have
\[
w \leq s_m \leq [\tilde{D} \circ \mu(\Gamma_{\text{id}})]^n(v).
\]

The term on the right-hand side does not depend on any particular choice of nesting of the index sets. Hence every \( w \) satisfying (21) also satisfies
\[
w \leq [\tilde{D} \circ \mu(\Gamma_{\text{id}})]^n(|v|_{\max}, \ldots, |v|_{\max})\theta
\]

and taking the maximum-norm on both sides yields
\[
|w|_{\max} \leq \phi(|v|_{\max})
\]

for some function \( \phi \) of class \( K_{\infty} \). For example, \( \phi \) can be chosen as
\[
\phi(t) := \max\{(\tilde{D} \circ \mu(\Gamma_{\text{id}}))^{\theta}(t, \ldots, t), ((\tilde{D} \circ \mu(\Gamma_{\text{id}}))^{\theta}(t, \ldots, t))_n\}.
\]

This completes the proof of the lemma.

IV. SMALL GAIN THEOREM

Now we turn back to the question of stability. In order to prove ISS of (6) we use the same approach as in [4]. The main idea is to prove that the system is GS and AG and then to use the result of [16] by which AG and GS systems are ISS.

So, let us prove at first small gain theorems for GS and AG of the system.

Theorem 4.1: Assume that each subsystem of (6) is GS. If there exists \( D \) as in (18) such that \( \Gamma \circ D(x) \not\leq x \) for all \( x \neq 0 \) is satisfied, then the system (1) is GS.

Proof: Let us take the supremum over \( t \in [0, \tau] \) on both sides of (9), (10). For \( i \in I_S \) we have
\[
\|x_{i[0,\tau]}\|_{\infty} \leq \sigma_i(|x_i(0)|) + \sum_{j=1}^{n} \gamma_{ij}(\|x_{j[0,\tau]}\|_{\infty}) + \gamma_i(\|u\|_{\infty})
\]

and for \( i \in I_{\text{max}} \) it follows
\[
\|x_{i[0,\tau]}\|_{\infty} \leq \max\{\sigma_i(|x_i(0)|), \max_j \gamma_{ij}(\|x_{j[0,\tau]}\|_{\infty})\}, \gamma_i(\|u\|_{\infty})
\]

(39)

(38)

Let us denote \( w = (\|x_{1[0,\tau]}\|_{\infty}, \ldots, \|x_{n[0,\tau]}\|_{\infty})\), \( (\Gamma)_{ij} = \gamma_{ij}, \)
\[
v = \left( \begin{array}{c}
\mu_{1}(\sigma_{1}(|x_{1}(0)|), \gamma_{1}(\|u\|_{\infty}) \\
\vdots \\
\mu_{n}(\sigma_{n}(|x_{n}(0)|), \gamma_{n}(\|u\|_{\infty}))
\end{array} \right)
\]

where we use notation \( \mu \) and \( \mu_\ast \) defined in (20). From (38), (39) we obtain \( w \leq \mu(\Gamma(w), v) \). Then by Lemma 3.2 there exists \( \phi \in K_{\infty} \) such that
\[
\|x_{i[0,\tau]}\|_{\infty} \leq \phi([\|\mu(\sigma(|x(0)|)\), \gamma(|\|u\|_{\infty})\|]) \leq \phi([\|\sigma(|x(0)|)\| + \gamma(|\|u\|_{\infty})\|] \leq \phi(2\|\sigma(|x(0)|)\| + \phi(2\|\gamma(|\|u\|_{\infty})\|))
\]

for some class \( K \) function \( \phi \) and all \( t > 0 \). Hence for every initial condition and essentially bounded input \( u \) the solution of the system (1) exists for all \( t \geq 0 \) and is uniformly bounded, since the right-hand side of (40) does not depend on \( t \). The estimate for GS is then given by (40).

Theorem 4.2: Assume that each subsystem of (6) has the AG property and that solutions of the system (1) exist for all positive times and are uniformly bounded. If there exists a \( D \) as in (18) such that \( \Gamma \circ D(x) \not\geq x \) for all \( x \neq 0 \), then system (1) satisfies the AG property.

Remark 4.3: The existence of solutions for all times is essential, otherwise the assertion is not true. See Example 14 in [4].

Proof: Let \( \tau \) be an arbitrary initial time. From the definition of the AG property we have for \( i \in I_S \)
\[
\lim_{t \to \infty} \sup_{t \in [0, \tau]} |x_i(t)| \leq \sum_{j=1}^{n} \gamma_{ij}(\|x_{j[\tau, \infty]}\|_{\infty}) + \gamma_i(\|u\|_{\infty})
\]

(41)

and for \( i \in I_{\text{max}} \)
\[
\lim_{t \to \infty} \sup_{t \in [0, \tau]} |x_i(t)| \leq \max\{\max_j \gamma_{ij}(\|x_{j[\tau, \infty]}\|_{\infty})\} \gamma_i(\|u\|_{\infty})
\]

(42)

Since all solutions of (6) are bounded Lemma 7 from [4] can be applied and we get that
\[
\lim_{t \to \infty} \sup_{t \in [0, \tau]} |x_i(t)| = \lim_{\tau \to \infty} \sup_{t \in [0, \tau]} |x_{i[\tau, \infty]}| =: l_i(x_i), i = 1, \ldots, n.
\]

By this property from (41), (42) and Lemma II.1 in [16] it follows that
\[
l_i(x_i) \leq \sum_{j=1}^{n} \gamma_{ij}(l_j(x_j)) + \gamma_i(\|u\|_{\infty})
\]

for \( i \in I_S \) and
\[
l_i(x_i) \leq \max\{\max_j \gamma_{ij}(l_j(x_j))\} \gamma_i(\|u\|_{\infty})
\]

for \( i \in I_{\text{max}} \). Using Lemma 3.2 we conclude
\[
\lim_{t \to \infty} \sup_{t \in [0, \tau]} |x(t)| \leq \phi(\|u\|_{\infty})
\]

(43)

for some \( \phi \) of class \( K \), which is desired AG property.

Theorem 4.4: Assume that each subsystem of (6) is ISS and \( \Gamma \) be defined by (13). If there exists a \( D \) as in (18) such that \( \Gamma \circ D(x) \not\geq x \) for all \( x \neq 0 \), then system (1) is ISS.

Proof: Since each subsystem is ISS it follows in particular that it is GS, see Theorem 1 in [16]. By Theorem 4.1 the whole interconnection (1) is then GS. This implies that solution of (1) exists for all times and is uniformly bounded.

Another consequence of ISS property of each subsystem is that each of them has the AG property. Applying Theorem 4.2 the whole system (1) has the AG property.

This implies that (1) is ISS by Theorem 1 in [16].
The following section gives a Lyapunov type counterpart of the small gain theorem obtained in this section and shows an explicit construction of an ISS Lyapunov function for interconnections of ISS systems.

V. CONSTRUCTION OF ISS LYAPUNOV FUNCTION

Again we consider an interconnection of $n$ subsystems in form of (6) where each subsystem is assumed to be ISS and hence there is a smooth ISS Lyapunov function for each subsystem. We will impose a small gain condition on the Lyapunov gains to prove the ISS property of the whole system (1) and we will look for an explicit construction of an ISS Lyapunov for it. For our purpose it is sufficient to work with not necessarily smooth Lyapunov functions defined as follows.

A continuous function $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, where $\alpha(t) = 0$ if and only if $t = 0$, is called proper and positive definite if there are $\psi_1, \psi_2 \in \mathcal{K}$ such that

$$\psi_1(||x||) \leq V(x) \leq \psi_2(||x||), \forall x \in \mathbb{R}^n.$$

**Definition 5.1:** A continuous function $V : \mathbb{R}^n \to \mathbb{R}^n$ is called an ISS Lyapunov function for the system (1) if

1) it is proper, positive definite and locally Lipschitz continuous on $\mathbb{R}^n \setminus \{0\}$

2) there exists $\gamma \in \mathcal{K}$, and a positive definite function $\alpha$ such that in all points of differentiability of $V$ we have

$$V(x) \geq \gamma(||u||) \Rightarrow \nabla V(x)f(x, u) \leq -\alpha(||u||). \quad (44)$$

Note that we do not require an ISS Lyapunov function to be smooth. However as a locally Lipschitz continuous it is differentiable almost everywhere.

**Remark 5.2:** In Theorem 2.3 in [7] it was proved that the system (1) is ISS if and only if it admits an (not necessary smooth) ISS Lyapunov function.

ISS Lyapunov function for subsystems can be defined in the following way.

**Definition 5.3:** A continuous function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}^n$ is called an ISS Lyapunov function for the subsystem $i$ in (6) if

1) it is proper and positive definite and locally Lipschitz continuous on $\mathbb{R}^{n_i} \setminus \{0\}$

2) there exist $\gamma_{ij} \in \mathcal{K}_\infty$, and a positive definite function $\alpha_i$ such that in all points of differentiability of $V_i$ we have

for $i \in I^\Sigma$

$$V_i(x_i) \geq \gamma_{i1}(V_i(x_1)) + \ldots + \gamma_{i n_i}(V_n(x_n)) + \gamma_i(||u||) \Rightarrow$$

$$\nabla V_i(x_i)f_i(x, u) \leq -\alpha_i(||x_i||) \quad (45)$$

and for $i \in I^\max$

$$V_i(x_i) \geq \max\{\gamma_{i1}(V_i(x_1)), \ldots, \gamma_{i n_i}(V_n(x_n)), \gamma_i(||u||)\} \Rightarrow$$

$$\nabla V_i(x_i)f_i(x, u) \leq -\alpha_i(||x_i||). \quad (46)$$

To construct an ISS Lyapunov function for the interconnected system (1) the notion of $\Omega$-path is needed.

**Definition 5.4:** A continuous path $\sigma \in \mathcal{K}_\infty$ is called an $\Omega$-path with respect to $\Gamma$ if

(i) for each $i$, the function $\sigma_i^{-1}$ is locally Lipschitz continuous on $(0, \infty)$;

(ii) for every compact set $P \subset (0, \infty)$ there are finite constants $0 < c < C$ such that for all points of differentiability of $\sigma_i^{-1}$ and $i = 1, \ldots, n$ we have

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C, \quad \forall r \in K \quad (47)$$

(iii) for all $r > 0$

$$\Gamma(\sigma(r)) < \sigma(r), \forall r > 0. \quad (48)$$

Let matrix $\Gamma$ be obtained from matrix $\Gamma$ by adding external gains $\gamma_i$ as the last column and let the map $\Gamma : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n_+$ be defined by:

$$\Gamma(s, r) := \{\Gamma_1(s, r), \ldots, \Gamma_n(s, r)\} \quad (49)$$

for $s \in \mathbb{R}^n_+$ and $r \in \mathbb{R}^n_+$, where $\Gamma_i : \mathbb{R}^{n_i+1}_+ \to \mathbb{R}^n_+$ is given by

$$\Gamma_i(s, r) := \gamma_{i1}(s_1) + \ldots + \gamma_{in}(s_n) + \gamma_i(r) \quad (i \in I^\Sigma) \quad \text{and by}$$

$$\Gamma_i(s, r) := \max\{\gamma_{i1}(s_1), \ldots, \gamma_{in}(s_n), \gamma_i(r) \quad (i \in I^\Sigma).$$

Before we proceed to the main result of this section let us recall a related result from [7] adapted for our situation:

**Theorem 5.5:** Consider the interconnection given by (1), (6) where each subsystem $i$ has an ISS Lyapunov function $V_i$ with the corresponding Lyapunov gains $\gamma_{ij}$, $\gamma_i$, $i, j = 1, \ldots, n$ as in (45) and (46). Let $\Gamma$ be defined as in (49). Assume that there is an $\Omega$-path $\sigma$ with respect to $\Gamma$ and a function $\phi \in \mathcal{K}_\infty$ such that

$$\Gamma(\sigma(r), \phi(r)) < \sigma(r), \quad \forall r > 0. \quad (50)$$

Then an ISS Lyapunov function for the overall system is given by

$$V(x) = \max_{i=1, \ldots, n} \sigma_i^{-1}(V_i(x_i)).$$

We note that this Theorem is a special case of Theorem 5.3 in [7] that was stated for a more general $\Omega$ then here. Moreover it was shown that an $\Omega$-path needed for the above construction always exists if $\Gamma$ is irreducible and $\Gamma \not\approx \text{id}$ in $\mathbb{R}_+^n$. The cases $I^\max = \emptyset$ and $I^\Sigma = \emptyset$ are particular cases of Corollary 5.5 and Corollary 5.6 from [7] respectively with irreducible $\Gamma$, where the existence of $\phi$ was shown under the condition $\Gamma \not\approx \text{id}$ in $\mathbb{R}_+^n$ in case of $I^\Sigma = \emptyset$. A stronger condition $D \circ \Gamma \not\approx \text{id}$ in $\mathbb{R}_+^n$ with $D = \text{diag}(\text{id} + \alpha)$ was used for $I^\max = \emptyset$.

The next result gives a counterpart of Corollaries 5.5 and 5.6 from [7] specified for the situation where both $I^\Sigma$ and $I^\max$ can be nonempty.

**Theorem 5.6:** Assume that each subsystem of (6) has an ISS Lyapunov function $V_i$ and the corresponding gain matrix is given by (49). If $\Gamma$ is irreducible and if there exists $D$ as in (18) such that $D \circ \Gamma \not\approx \text{id}$ for all $x \neq 0$ is satisfied, then the system (1) is ISS and an ISS Lyapunov function is given by

$$V(x) = \max_{i=1, \ldots, n} \sigma_i^{-1}(V_i(x_i)), \quad (51)$$

where $\sigma \in \mathcal{K}_\infty$ is an arbitrary $\Omega$-path with respect to $D \circ \Gamma$.

**Proof:** The existence of an $\Omega$-path $\sigma$ follows from Theorem 5.2(ii) in [7] applied on $\Gamma := D \circ \Gamma$, i.e., there are $\sigma_i \in \mathcal{K}_\infty$ such that $\sigma = (\sigma_1, \ldots, \sigma_n)^T$ satisfies

$$D \circ \Gamma(\sigma) < \sigma.$$
From the structure of $D$ it follows that
\[
\sigma_i > (i\alpha + \alpha) \circ \Gamma_i(\sigma), \quad i \in I_{\Sigma},
\]
\[
\sigma_i > \Gamma_i(\sigma), \quad i \in I_{\max}.
\]

The irreducibility of $\Gamma$ assures that $\Gamma(\sigma)$ is unbounded in all components. Let $\phi \in \mathcal{K}_\infty$ be such that for all $r \geq 0$ the inequality $\alpha(\Gamma_1(\sigma(r))) \geq \max_{i=1,\ldots,n} \gamma_i(\phi(\sigma))$ holds for $i \in I_{\Sigma}$ and $\Gamma_i(\sigma(\sigma)) \geq \max_{i=1,\ldots,n} \gamma_i(\phi(\sigma))$ for $i \in I_{\max}$. Note that such $\phi$ always exists and can be chosen, for example, as follows. For any $\gamma_i \in \mathcal{K}$ we chose $\tilde{\gamma_i} \in \mathcal{K}_\infty$ such that $\tilde{\gamma_i} \geq \gamma_i$. Then $\phi$ can be taken as $\phi(r) := \frac{1}{2} \min_{i \in I_{\Sigma}, j \in I} \tilde{\gamma_j}^{-1}(\alpha(\Gamma_i(\sigma(r))))$, $\min_{i \in I_{\max}, j \in I} \tilde{\gamma_j}^{-1}(\Gamma_i(\sigma(r)))$.

Note that $\phi$ is a $\mathcal{K}_\infty$ function since minimum over $\mathcal{K}_\infty$ functions is again of class $\mathcal{K}_\infty$. Then we have for all $r > 0, i \in I_{\Sigma}$ that
\[
\sigma_i(r) > D_{ij} \circ \Gamma_i(\sigma(r)) = \Gamma_i(\sigma(r)) + \alpha(\Gamma_i(\sigma(r))) \geq \Gamma_i(\sigma(r)) + \gamma_i(\phi(\sigma)) = \Gamma_i(\sigma(r), \phi(\sigma))
\]
and for all $r > 0, i \in I_{\max}$
\[
\sigma_i(r) > D_{ij} \circ \Gamma_i(\sigma(r)) = \Gamma_i(\sigma(r)) \geq \max\{\Gamma_i(\sigma(r)), \gamma_i(\phi(\sigma))\} = \Gamma_i(\sigma(r), \phi(\sigma))
\]
Thus $\sigma(r) > \Gamma(\sigma(r), \phi(\sigma))$ and the assertion follows from Theorem 5.5.

Irreducibility of $\Gamma$ assumed in Theorem 5.6 means in particular that the graph representing the interconnection structure of the whole system is strongly connected. However in general this graph can be not strongly connected.

The matrix $\Gamma$ is reducible in this case. A Lyapunov function can be constructed also in this case. The construction is based on the construction for irreducible case. From [2] if matrix is reducible it can be transformed to the upper block triangular form via a permutation. I.e., after a renumbering of the subsystems, the matrix $\Gamma$ is in the form of:
\[
\Gamma = \begin{pmatrix}
\Gamma_{11} & \Gamma_{12} & \ldots & \Gamma_{1d} & \Gamma_1 \\
0 & \Gamma_{22} & \ldots & \Gamma_{2d} & \Gamma_2 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & \Gamma_{dd} & \Gamma_d
\end{pmatrix}
\]
\[\text{(52)}\]

where each block on the diagonal $\Gamma_{jj} \in (\mathcal{K}_\infty \cup \{0\})^{d_j \times d_j}, j = 1, \ldots, d$, is either irreducible or 0.

The following theorem can be used for construction of ISS Lyapunov functions. This result is based on Corollary 6.3 and Corollary 6.4 in [7].

**Theorem 5.7:** Assume that each subsystem of (6) has an ISS Lyapunov function $V_i$ and the corresponding gain matrix is given by (49). If there exists $D$ as in (18) such that $\Gamma \circ D(x) \nless x$ for all $x \neq 0$ is satisfied, then the system (1) is ISS, moreover there exists an $\Omega$-path $\sigma$ and $\phi \in \mathcal{K}_\infty$ satisfying $\Gamma(\sigma, \phi) < \sigma$ and an ISS Lyapunov function for the whole system (1) can be taken as
\[V(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i)).\]

**Proof:** After a renumbering of subsystems we can assume that $\Gamma$ is of the form (52). And let $D$ be the corresponding diagonal operator that contains $i\alpha$ or $i\alpha + \alpha$ on the diagonal depending on the new numeration of the subsystems. Let the state $x$ be partitioned into $z_i \in \mathbb{R}^{d_i}$ where $d_i$ is the size of the $i$th diagonal block $\mathcal{Y}_{ii}$, $i = 1, \ldots, d$. And consider the subsystems $\Sigma_j$ of the whole system (1) with these states
\[z_j := (x_{q_j+1}^T, x_{q_j+2}^T, \ldots, x_{q_{j+1}}^T),\]
where $q_j = \sum_{i=1}^{j-1} d_i$, with the convention that $q_0 = 0$. Note that each $\mathcal{Y}_{jj}, j = 1, \ldots, n$ satisfies the small gain condition of the form $\mathcal{Y}_{jj} \circ D_j \nless \id$ where $D_j : \mathbb{R}^{d_i} \to \mathbb{R}^{d_j}$ is a corresponding part of $D$.

For each $\Sigma_j$ with the gain operator $\mathcal{Y}_{jj}, j = 1, \ldots, n$ and external inputs $z_{j+1}, \ldots, d$, in Theorem 5.6 is applicable and implies that there is an ISS Lyapunov function $W_j = \max_{i=q_j+1, \ldots, q_{j+1}} \tilde{\gamma_i}^{-1}(V_i(x_i))$ for $\Sigma_j$, where $\tilde{\gamma_i} := \alpha(\Gamma_1(\sigma(r)))$, $i = q_j+1, \ldots, q_{j+1}$ and $\gamma_i := \alpha(\Gamma_1(\sigma(r)))$. By induction over the number of blocks it can be proved that an ISS Lyapunov function for the whole system (1) exists and is of the form $\bar{V}(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i))$, where $\sigma$ is an arbitrary $\Omega$-path with respect to $\Gamma \circ D$:

For $d = 1$ there is nothing to show. Assume that for the system corresponding to $d = k-1$ blocks an ISS Lyapunov function $\bar{V}_{k-1}$ exists and is equal to $\bar{V}_{k-1} = \max_{i=1,\ldots,q_k} \sigma_i^{-1}(V_i(x_i))$. Consider now $d = k$ blocks. By assumption for the system corresponding to the first $k-1$ blocks there exists an ISS Lyapunov function $\bar{V}_{k-1}$, where $\bar{z}_{k-1} := (z_1, \ldots, z_{k-1})^T$ is its state. This means that
\[\bar{V}_{k-1}(\bar{z}_{k-1}) \geq \tilde{\gamma}_{k-1,k}(W_k(z_k)) + \tilde{\gamma}_{k-1,u}(||u||) \Rightarrow \]
\[\nabla \bar{V}_{k-1}(\bar{z}_{k-1}) f_{k-1}(\bar{z}_{k-1}, z_k, u) \leq -\tilde{\alpha}_{k-1}(||\bar{z}_{k-1}||)\]

where $\tilde{\gamma}_{k-1,k}, \tilde{\gamma}_{k-1,u}$ are the corresponding gains, $f_{k-1}, \tilde{\alpha}_{k-1} \bar{z}$ are the corresponding derivative of the state and the estimate of Lyapunov derivative.

As the system corresponding to the block $k$ has only external input with a gain $\gamma_{k,u}$ the matrix $\bar{\Gamma}$ is of the form
\[\bar{\Gamma} = \begin{pmatrix} 0 & \tilde{\gamma}_{k-1,k} & \tilde{\gamma}_{k-1,u} \\ 0 & \gamma_{k,u} \end{pmatrix}.\]

For such $\bar{\Gamma}$ by Lemma 6.1 in [7] there exists an $\Omega$-path $\tilde{\sigma}^k = (\tilde{\sigma}_1^k, \tilde{\sigma}_2^k)^T \in \mathcal{K}_\infty$ and $\phi \in \mathcal{K}_\infty$ such that $\bar{\Gamma}(\tilde{\sigma}_1^k, \phi) < \bar{\sigma}^k$ holds. Applying Theorem 5.5 an ISS Lyapunov function for
the whole system exists and is given by
\[
\tilde{V}_k = \max\{\tilde{\sigma}_k^{-1}(\tilde{V}_{k-1}), (\tilde{\sigma}_k^{-1})(W_k)\} = \\
\max\{\tilde{\sigma}_1^{-1} \circ \max_{i=1,\ldots,q_k} \sigma_i^{-1}(V_i(x_i)), \\
(\tilde{\sigma}_k^{-1}) \circ \max_{i=q_k+1,\ldots,q_{k+1}} \sigma_i^{-1}(V_i(x_i))\} = \\
\max\{\max_{i=1,\ldots,q_k} \sigma_i \circ (\tilde{\sigma}_k)^{-1}(V_i(x_i)), \\
\max_{i=q_k+1,\ldots,q_{k+1}} (\tilde{\sigma}_i \circ (\tilde{\sigma}_k)^{-1}(V_i(x_i)))\} = \\
\max\{\max_{i=1,\ldots,q_k} \sigma_i, \max_{i=q_k+1,\ldots,q_{k+1}} \tilde{\sigma}_i\}(\tilde{\sigma}_k)^{-1}(V_i(x_i)), \tag{53}
\]
where \(\sigma_i = \sigma_i \circ \tilde{\sigma}_i \in K_\infty\) for \(i = 1,\ldots,q_k\) and \(\tilde{\sigma}_i = \tilde{\sigma}_i \circ \tilde{\sigma}_k \in K_\infty\) for \(i = q_k + 1,\ldots,q_{k+1}\).

Thereby the assertion of the induction is proved and the whole system (1) has an ISS Lyapunov function given by
\[
V(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i)), \quad \sigma \in \Omega \text{ with respect to } \Gamma \circ \mathcal{D}.
\]

\section{VI. Example}

To demonstrate the advantages of Theorem 4.4 we consider the interconnection system (6) with \(n = 3\) where each subsystem is ISS:
\[
\begin{align*}
|x_1(t)| &\leq \beta_1(|x(0)|) + \gamma_{13}(|x_{30}(t)|) + \gamma_{1}(||u||_{\infty}) \\
|x_2(t)| &\leq \max\{\beta_2(|x(0)|), \gamma_{21}(|x_{10}(t)|)\}, \quad \gamma_{23}(||x_{30}(t)||_{\infty}), \quad \gamma_{2}(||u||_{\infty}) \\
|x_3(t)| &\leq \max\{\beta_3(|x(0)|), \gamma_{32}(|x_{20}(t)||_{\infty}), \gamma_{3}(||u||_{\infty})\} \tag{54}
\end{align*}
\]
with gains given by \(\gamma_{13}(t) = (id + \rho)^{-1}(t), \rho \in K_\infty\), \(\gamma_{21}(t) = t, \gamma_{23}(t) = t\) and \(\gamma_{32}(t) = t(1 - e^{-t}), t \geq 0\). In this case we have the following
\[
\Gamma = \begin{pmatrix}
0 & 0 & \gamma_{13} \\
\gamma_{21} & 0 & \gamma_{23} \\
0 & \gamma_{32} & 0
\end{pmatrix}
\]
Then the small gain condition (19) becomes
\[
\begin{pmatrix}
\gamma_{13}(s_3) \\
\max\{\gamma_{21} \circ (id + \alpha)(s_1), \gamma_{23}(s_3)\} \\
\gamma_{32}(s_2)
\end{pmatrix} \not\preceq \begin{pmatrix}
s_1 \\
s_2 \\
s_3
\end{pmatrix}
\]
for all \(s \in \mathbb{R}_+^3 \setminus \{0\}, t \geq 0\). This condition is equivalent to
\[
(id + \alpha) \circ \gamma_{13} \circ \gamma_{32} \circ \gamma_{21}(t) < t \tag{55}
\]
and simultaneously
\[
\gamma_{23} \circ \gamma_{32}(t) < t \tag{56}
\]
for all \(t > 0\).

For \(\alpha = \rho\) the inequality (55) is satisfied:
\[
(id + \alpha) \circ (id + \rho)^{-1} \circ (t(1 - e^{-t})) \tag{57}
\]
and simultaneously
\[
\gamma_{23} \circ \gamma_{32}(t) < t \tag{58}
\]
for all \(t > 0\).

The inequality (56) is also satisfied:
\[
t(1 - e^{-t}) < t.
\]

Then by Theorem 4.4 we conclude that system (1) is ISS. We would like to point out that application of the small gain condition from [4] will not help us to prove stability for this example.

In order to apply results from [4] we need to use (17) in (54). Then we obtain estimations of trajectories by
\[
\begin{align*}
|x_1(t)| &\leq \beta_1(|x(0)|) + \gamma_{13}(|x_{30}(t)|) + \gamma_{1}(||u||_{\infty}) \\
|x_2(t)| &\leq \max\{\beta_2(|x(0)|), \gamma_{21}(|x_{10}(t)|), \gamma_{23}(|x_{30}(t)|) + \gamma_{2}(||u||_{\infty})\} \\
|x_3(t)| &\leq \max\{\beta_3(|x(0)|), \gamma_{32}(|x_{20}(t)|| + \gamma_{3}(||u||_{\infty})\} \tag{59}
\end{align*}
\]
The small gain condition from [4] is: there exist \(\alpha_i \in K_\infty\), \(i = 1, 2, 3\) such that
\[
\begin{pmatrix}
\gamma_{13} \circ (id + \alpha)(s_3) \\
\gamma_{21} \circ (id + \alpha)(s_1) + \gamma_{23} \circ (id + \alpha)(s_3) \\
\gamma_{32} \circ (id + \alpha)(s_2)
\end{pmatrix} \not\preceq \begin{pmatrix}
s_1 \\
s_2 \\
s_3
\end{pmatrix}
\]
for all \(s \in \mathbb{R}_+^3 \setminus \{0\}\).

This condition implies for \(s = \left(\begin{array}{c}
0 \\
t \\
\gamma_{32} \circ (id + \alpha)(t)
\end{array}\right)\) and \(t \geq 0\), that
\[
\gamma_{23} \circ (id + \alpha) \circ \gamma_{32}(t) < t
\]
or the even weaker condition
\[
\gamma_{23} \circ (id + \alpha) \circ \gamma_{32}(t) < t \tag{59}
\]
Suppose such an \(\alpha_2\) exists. Then
\[
\gamma_{23} \circ (id + \alpha_2) \circ \gamma_{32}(t) = (id + \alpha_2)(t(1 - e^{-t})) < t.
\]
Hence,
\[
(t(1 - e^{-t})) + \alpha_2 t(1 - e^{-t}) < t \quad \text{or} \quad \alpha_2 t(1 - e^{-t}) < te^{-t}.
\]
This leads to a contradiction, since \(\lim_{t \to \infty} \alpha_2(t(1 - e^{-t})) = +\infty\) and \(\lim_{t \to \infty} te^{-t} = 0\). It follows that there are no \(\alpha_2 \in K_\infty\) such that (59) is satisfied. So by this small gain condition we cannot conclude whether the interconnection is ISS.

A similar argument applies, if the ISS formulation is transformed to the maximum formulation throughout. Again, an inconclusive formulation results.

This example shows the sharpness of the refined small gain condition developed in this paper for the case of differently characterized ISS systems.

\section{VII. Conclusion}

We have considered several ISS systems in an interconnection. The gains of these systems are defined in two different ways. This kind of interconnections is more general then in [4]. In case if an interconnection contains differently characterized ISS subsystems the known small gain results can be applied with costs of certain conservativeness. A less conservative small gain condition assuring the ISS property of such interconnection is proved in this paper. An example shows the effectiveness and advantage of this condition in comparison to known results.
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