Haar Wavelet-like Analysis with MRA Method Extended to Fractals

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Sept 18, 2012

Outline

1. Signal and Image processing,
2. Selfsimilarity, Computational Features
3. Slanted Matrix Representations
4. Image Decomposition using Forward Wavelet Transform
5. Wavelets and Fractals, and Fractal image processing

Signal and Image processing

(a) A systematic study of bases in Hilbert spaces built on fractals suggests a common theme: A hierarchical multiscale structure. A well-known instance of the self-similarity is reflected in the scaling rules from wavelet theory.

(b) The best known instances of this are the dyadic wavelets in $L^2(\mathbb{R})$. They are built by two functions φ and ψ; subject to the relation

$$\varphi(x) = 2\sum h_n \varphi(2x - n), \text{ and } \psi(x) = 2\sum g_n \varphi(2x - n). \tag{1}$$

where (h_n) and (g_n) are fixed and carefully chosen sequences.

(c) The function φ is called the scaling function, or the father function, and ψ is called the mother function.

Signal and Image processing - cont'd

(d) The best known choice of pairs of filter coefficients (h_n), (g_n) is the following: Pick $(h_n) \subset \mathbb{R}$ subject to the two conditions $\sum_{n\in\mathbb{Z}} h_n = 1$ and $\sum_{n\in\mathbb{Z}} h_n h_{n+2l} = \frac{1}{2}\delta_{0,l}$. Then set $g_n := (-1)^n \overline{h}_{1-n}$, $n \in \mathbb{Z}$.

(e) The convention is that (h_n) is 0 outside some specified range.

(f) The associated double indexed family $\psi_{jk}(x) = 2^{i/2}\psi(2^j x - k)$, $j, k, \in \mathbb{Z}$ will be a wavelet basis for $L^2(\mathbb{R})$.

Signal and Image processing - cont'd

(g) This is the best known wavelet construction also known by the name multi-resolution analysis (MRA). The reason for this is that the father function φ generates a subspace V_0 of $L^2(\mathbb{R})$ which represents a choice of resolution for wavelet decomposition.

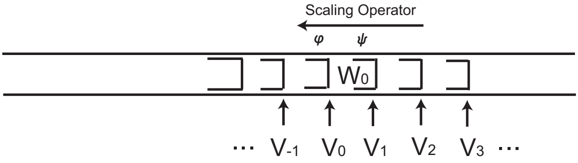


Figure: Multiresolution. $L^2(\mathbb{R}^d)$-version (continuous); $\varphi \in V_0$, $\psi \in W_0$.

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots, V_0 + W_0 = V_1.$$

Signal and Image processing - cont'd

Definition

Take V_0 to be the closed span of all the translates $(\varphi(\cdot - k))$, $k \in \mathbb{Z}$ in $L^2(\mathbb{R})$. From (1), it follows that the scaling operator $Uf(x) = 2^{-1/2} f(\frac{x}{2})$ maps the space V_0 into itself; and that $U\psi \in V_0$.

With suitable modification this idea also works for wavelet bases in $L^2(\mathbb{R}^d)$, and in Hilbert spaces built on fractals.

Selfsimilarity

(a) For Julia sets X in complex analysis for example, U could be implemented by a rational function $z \mapsto r(z)$.

(b) When r is given, X will be a compact subset of $\mathbb{C}$ which is determined by the dynamics of $r^n = \underbrace{r \circ \cdots \circ r}_{n \text{ times}}$. Specifically,

$$\mathbb{C} \backslash X = \cup \{ \mathcal{O} | \mathcal{O} \text{ open, } (r^{(n)} | \mathcal{O}) \text{ is normal} \}. \tag{2}$$

(c) Interested in showing that these non-linear fractals are related to more traditional wavelets, i.e., those of $L^2(\mathbb{R}^d)$. We want to extend the $\mathbb{R}^d$-analysis both to fractals and to discrete hierarchical models.

Computational Features

(a) Approximation of the father or mother functions by subdivision schemes.

(b) Matrix formulas for the wavelet coefficients. A variety of data will be considered; typically for fractals, L^2-convergence is more restrictive than is the case for $L^2(\mathbb{R}^d)$-wavelets.

This makes wavelets closely related to fractals and fractal processes. A unifying approach to wavelets, dynamical systems, iterated function systems, self-similarity and fractals may be based on the systematic use of operator analysis and representation theory.

Operator Theoretic Models

(a) Motivation: hierarchical models and multiscaling, operators of multiplication, and dilations, and more general weighted composition operators are studied. Scaling is implemented by non-linear and non-invertible transformations. This generalizes affine trasformations of variables from wavelet analysis and analysis on affine fractals.

(b) The properties of dynamical and iterated function systems, defined by these transformations, govern the spectral properties and corresponding subspace decompositions.

Operator Theoretic Models - cont'd

(c) The interplay between dynamical and iterated function systems and actions of groups and semigroups on one side, and operator algebras on the other side, yield new results and methods for wavelets and fractal analysis and geometry.

(d) Wavelets, signals and information may be realized as vectors in a real or complex Hilbert space. In the case of images, this may be worked out using wavelet and filter functions, e.g. corresponding to ordinary Cantor fractal subsets of $\mathbb{R}$, as well as for fractal measure spaces of Sierpinski Gasket fractals.

Operators and Hilbert Space

(a) Operator algebra constructions of covariant representations are used in the analysis of orthogonality in wavelet theory, in the construction of super-wavelets, and orthogonal Fourier bases for affine fractal measures.

(b) In signal processing, time-series, or matrices of pixel numbers may similarly be realized by vectors in Hilbert space $\mathcal{H}$.

(c) In signal/image processing, because of aliasing, it is practical to generalize the notion of ONB, and this takes the form of what is “a system of frame vectors.”

Operators and Hilbert Space - cont'd

(d) One particular such ONB goes under the name "the Karhunen-Loève basis."

(e) Motivation comes from the consideration of the optimal choices of bases for certain analogue-to-digital (A-to-D) problems we encountered in the use of wavelet bases in image-processing.

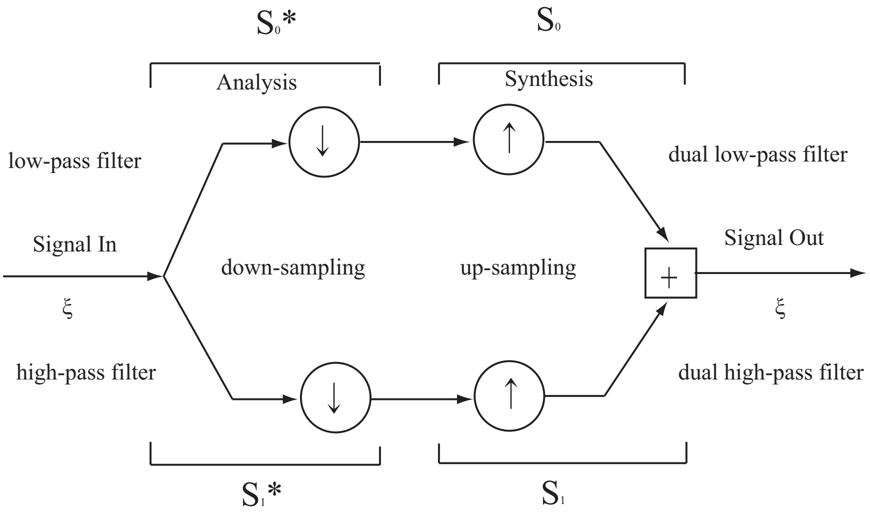
Definition

For every finite n, a representation of the Cuntz algebra $\mathcal{O}_n$ is a system of isometries $S_i : \mathcal{H} \to \mathcal{H}$ such that

(a) $S_i^* S_j = \delta_{ij} I$; orthogonality, and

(b) $\sum_i S_i S_i^* = I$ (perfect reconstruction).

Operators and Hilbert Space - cont'd



Slanted Matrix Representations

Definition

If $(h_n)_{n\in\mathbb{Z}}$ is a double infinite sequence of complex numbers, i.e., $h_n \in \mathbb{C}$, for all $n \in \mathbb{Z}$; set

$$(S_0 x)(m) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_{m-2n} x(n) \tag{3}$$

and adjoint

$$(S_0^* x)(m) = \sqrt{2} \sum_{n\in\mathbb{Z}} \bar{h}_{n-2m} x(n); \text{ for all } m \in \mathbb{Z}. \tag{4}$$

Slanted Matrix Representations - cont'd

Then

(a) The $\infty \times \infty$ matrix representations (3) and (4) have the following slanted forms

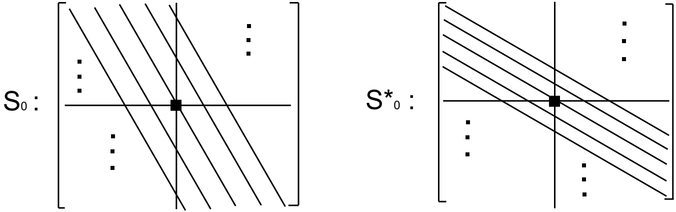


Figure: S_0 and S_0^*.

Slanted Matrix Representations - cont'd

(b) The set of non-zero numbers in $(h_n)_{n\in\mathbb{Z}}$ is finite if and only if the two matrices in Figure are *banded*.

(c) Relative to the inner product

$$\langle x|y\rangle_{l^2} := \sum_{n\in F} \bar{x}_n y_n \text{ in } l^2$$

(i.e., conjugate-linear in the first variable), the operator S_0 is *isometric* if and only if

$$\sum_{n\in F} \bar{h}_n h_{n+2p} = \frac{1}{2}\delta_{0,p}, \text{ for all } p \in \mathbb{Z}. \tag{5}$$

Slanted Matrix Representations - cont'd

(d) If (5) holds, and

$$(S_1 x)(m) = \sqrt{2} \sum_{n \in F} g_{m-2n} x(n), \tag{6}$$

then

$$S_0 S_0^* + S_1 S_1^* = I_{l^2} \tag{7}$$

$$S_k^* S_l = \delta_{k,l} I_{l^2} \text{ for all } k, l \in \{0, 1\} \tag{8}$$

(the Cuntz relations) holds for

$$g_n := (-1)^n \bar{h}_{1-n}, \ n \in \mathbb{Z}.$$

Slanted Matrix Representations - cont'd

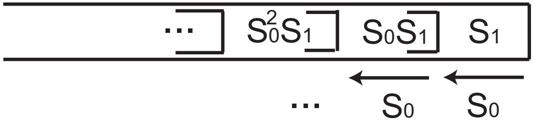


Figure: Multiresolution. $l^2(\mathbb{Z})$-version (discrete); $\varphi \in V_0$, $\psi \in W_0$.

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots, V_0 + W_0 = V_1.$$

Image Decomposition using Forward Wavelet Transform

A 1-level wavelet transform of an $N \times M$ image can be represented as

$$\mathbf{f} \mapsto \begin{pmatrix} \mathbf{a}^1 & | & \mathbf{h}^1 \\ -- & & -- \\ \mathbf{v}^1 & | & \mathbf{d}^1 \end{pmatrix}$$

where the subimages $\mathbf{h}^1, \mathbf{d}^1, \mathbf{a}^1$ and $\mathbf{v}^1$ each have the dimension of $N/2$ by $M/2$.

Image Decomposition using Forward Wavelet Transform-cont'd

$$\mathbf{a}^1 = V_m^1 \otimes V_n^1 : \varphi^A(x,y) = \varphi(x)\varphi(y)$$
$$= \sum_i \sum_j h_i h_j \varphi(2x-i)\varphi(2y-j)$$
$$\mathbf{h}^1 = W_m^1 \otimes V_n^1 : \psi^H(x,y) = \psi(x)\varphi(y)$$
$$= \sum_i \sum_j g_i h_j \varphi(2x-i)\varphi(2y-j)$$
$$\mathbf{v}^1 = V_m^1 \otimes W_n^1 : \psi^V(x,y) = \varphi(x)\psi(y)$$
$$= \sum_i \sum_j h_i g_j \varphi(2x-i)\varphi(2y-j)$$
$$\mathbf{d}^1 = W_m^1 \otimes W_n^1 : \psi^D(x,y) = \psi(x)\psi(y)$$
$$= \sum_i \sum_j g_i g_j \varphi(2x-i)\varphi(2y-j)$$

Image Decomposition using Forward Wavelet Transform-cont'd

φ : the father function in sense of wavelet.

ψ : is the mother function in sense of wavelet.

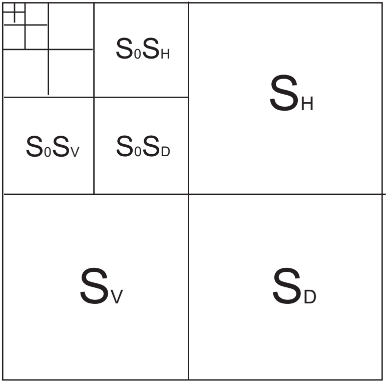
V space : the average space and the from multiresolution analysis (MRA).

W space : the difference space from MRA.

h : low-pass filter coefficients

g : high-pass filter coefficients.

Subdivided Squares



Subdivided Squares-cont'd

(a) The subdivided squares represent the use of the pyramid subdivision algorithm used on pixel squares.

(b) At each subdivision step the top left-hand square represents averages of pixel values, averages taken with respect to the chosen low-pass filter; and the rest, horizontal, vertical, and diagonal detail differences, represented by separate bands and filters.

(c) So there are four bands, and they may be realized by a tensor product construction applied to dyadic filters in the separate $x-$ and the $y-$directions in the plane.

(d) The iteration of four isometries $S_0, S_H, S_V, and S_D$ with mutually orthogonal ranges, and satisfying the following sum-rule $S_0S_0^* + S_HS_H^* + S_VS_V^* + S_DS_D^* = I$, with I denoting the identity operator in an appropriate l^2-space are used.

Test Image

Figure: Prof. Jorgensen in his office.

First-level Decomposition

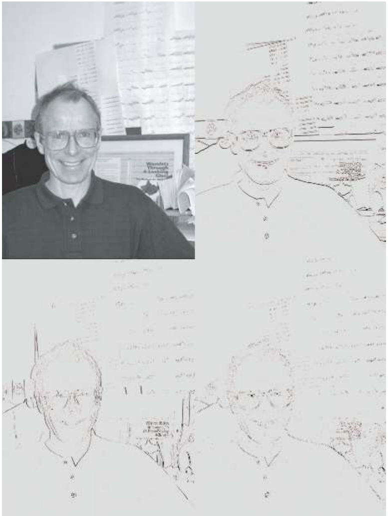


Figure: 1-level Haar Wavelet Decomposition of Prof. Jorgensen

Second-level Decomposition

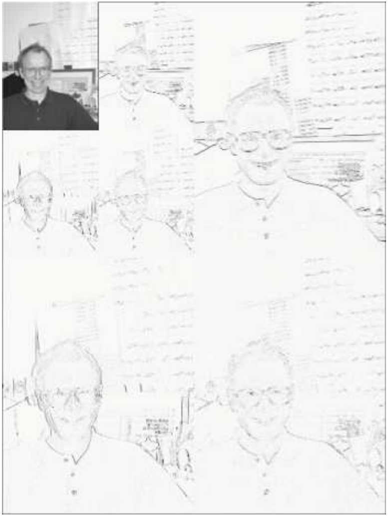
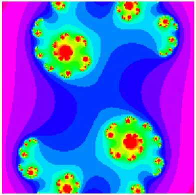
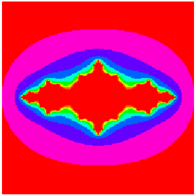


Figure: 2-level Haar Wavelet Decomposition of Prof. Jorgensen

Wavelets and Fractals





(a) The simplest Julia sets come from a one parameter family of quadratic polynomials $\varphi_c(z) = z^2 + c$, where z is a complex variable and where c is a fixed parameter.

(b) Consider the two branches of the inverse $\beta_\pm = z \mapsto \pm\sqrt{z - c}$. Then J_c is the unique minimal non-empty compact subset of $\mathbb{C}$, which is invariant under $\{\beta_\pm\}$.

(c) Interested in adapting and modifying the Haar wavelet, and the other wavelet algorithms to the Julia sets.

Wavelets and Fractals - cont'd

(d) There was an initition of wavelet transforms for complex fractals.

(e) These transforms have some parallels to traditional affine fractals, but subtle non-linearities precluded from writing down an analogue of Haar wavelets in these different settings.

(f) Want to develop more refined algorithms taking these difficulties into account.

Fractal Image Processing

(a) Unlike wavelets, fractal coders store images as a fixed points of maps on the plane instead of a set of quantized transform coefficients.

(b) Fractal compression is related to vector quantization, but fractal coders use a self-referential vector codebook, drawn from the image itself, instead of a fixed codebook.

(c) IFS theory motivates a broad class fractal compression schemes but it does not show why particular fractal schemes work well.

(d) A wavelet-based framework for analyzing fractal block coders would simplify the analysis of these codes considerably and give a clear picture of why they are effective.

References

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