Shift-Invariant and Sampling Spaces in the Fractional Fourier Transform Domain

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- It was implicitly used by E. Condon in 1937, *Proc. National Academy of Science.*
- But it turned out that N. Wiener actually introduced it in *J. Math. Phys. MIT,* (1929) as a way to solve certain classes of ordinary and partial differential equations arising in quantum mechanics.

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Some of the early pioneers in the field are L. Almeida, M. Kutay, A. Lohmann, D. Mendlovic, D. Mustard, H. Ozaktas, and Z. Zalevsky. Journals of IEEE , and Opt. Soc. Amer., and Australian Math. Soc.
The Fractional Fourier Transform with Applications in Optics and Signal Processing, H. Ozaktas, Z. Zalevsky, and M. Kutay, Wiley (2001)

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- A graded-index or gradient-index fiber is an optical fiber whose core has a refractive index that decreases with increasing radial distance from the fiber axis (the imaginary central axis running down the length of the fiber).

- Because parts of the core closer to the fiber axis have a higher refractive index than the parts near the cladding, light rays follow sinusoidal paths down the fiber. The advantage of the graded-index fiber compared to multimode step-index fiber is the decrease in modal dispersion.

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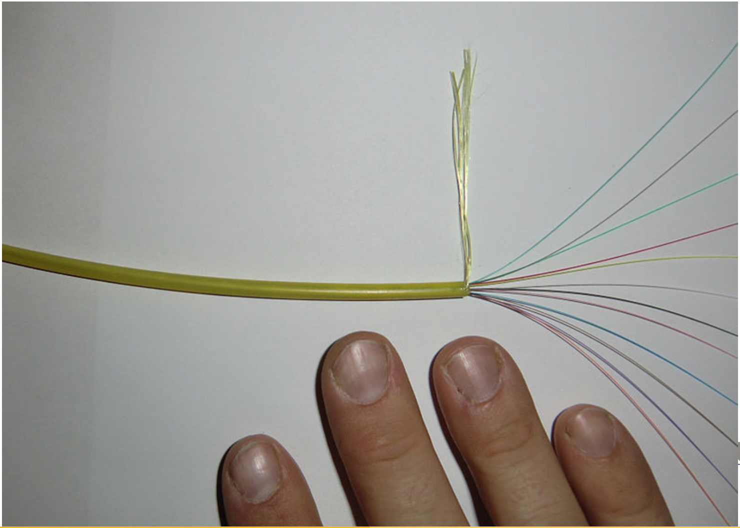
Graphics



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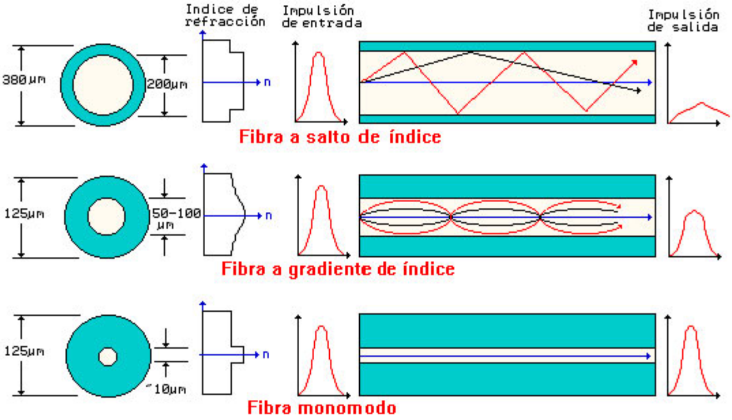
Indice de refracción
Impulsión de entrada
Impulsión de salida
380µm
200µm
n
Fibra a salto de índice
125µm
50–100 µm
n
Fibra a gradiente de índice
125µm
10µm
n
Fibra monomodo

For light propagation in quadratic graded-index media (fiber optics), it is known that the Fourier transform is produced at a certain distance d_0 that depends on the medium. Thus, it is reasonable to call the light distribution at distance $ad_0, 0 < a \leq 1$, the fractional Fourier transform of order a.

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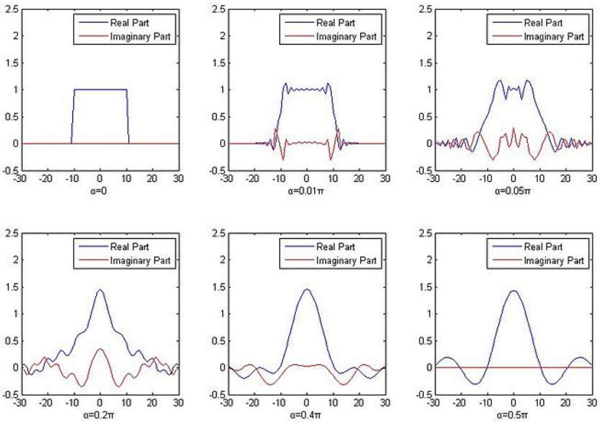

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- It was introduced into signal analysis by Ville "*Theorie et applications de la notion de signal analytique,*" **Cables et Transmissions,** Vol.2A, pp.61-74, (1948).
- The Wigner (Wigner-Ville) distribution (WVD) considers an analytic version of the signal for overcoming the cross terms generated by the negative spectra. It presents a unique signature of the signal possesses all the desirable properties of a time-frequency representation.

Definition

The Wigner distribution (WD) $W_{f,g}(t,\omega)$, of two signals f and g is defined by

$$W_{f,g}(t,\omega) = \int f\left(t + \frac{1}{2}\tau\right)\bar{g}\left(t - \frac{1}{2}\tau\right)e^{-2\pi i\omega\tau}d\tau.$$

Moyal's formula holds

$$\langle W_{f_1,g_1}, W_{f_2,g_2}\rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2\rangle_{L^2(\mathbb{R})}\overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R})}. \quad (1)$$

In particular,

$$\left\| W_{f,g} \right\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}, \quad (2)$$

The Wigner distribution is closely related to the cross-ambiguity function $A_{f,g}(u,v)$ of two functions f, g which is defined as

$$A_{f,g}(u,v) = \int f(t+u/2)\,\bar{g}(t-u/2)\,e^{-2\pi i v t} dt.$$

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The function $R_{f,g}(u)=A_{f,g}\left(u,0\right)$ is called the cross-correlation function of f and g and $R_f(u)=A_f\left(u,0\right)$ is called the auto-correlation function of f.

The Radar Ambiguity function is defined as

$$A(t,\omega)=\int f(\tau)\bar{f}(\tau+t)e^{-2\pi i\omega\tau}d\tau.$$

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What does correspond to a rotation by an angle $\pi/4$? Whatever it is, we call it the one half Fourier transform.

More generally, what does correspond to a rotation by an angle θ? ,i.e., Find g such that

$$W_g(u, v) = W_f(u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta).$$

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g is the fractional Fourier transform with angle θ.

The Fractional Fourier transform may also be viewed as a (family of bounded operators) $\mathcal{F}_\alpha$, with $0 \leq \alpha \leq 1$, such that

$$\mathcal{F}_0(f) = f, \quad \mathcal{F}_1 = \hat{f}.$$

The Fractional Fourier transform may also be viewed as a (family of bounded operators) $\mathcal{F}_\alpha$, with $0 \leq \alpha \leq 1$, such that

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In practice, it is indexed by an angle $0 \leq \theta \leq 2\pi$ so that

$$\mathcal{F}_0(f) = f, \quad \mathcal{F}_{\pi/2} = \hat{f}, \quad \mathcal{F}_\pi\left(f(x)\right) = f(-x), \quad \mathcal{F}_{2\pi} = f.$$

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$$F_{\theta}[f](\omega) = \widehat{f}_{\theta}(\omega) = \int_{-\infty}^{\infty} f(t)\mathcal{K}_{\theta}(t,\omega)\,dt \qquad (3)$$

where

$$\mathcal{K}_{\theta}(t,\omega) = \begin{cases} c(\theta)\cdot e^{ja(\theta)(t^2+\omega^2)-jb(\theta)\omega t}, & \theta \neq p\pi \\ \delta(t-\omega), & \theta = 2p\pi \\ \delta(t+\omega), & \theta = (2p-1)\pi \end{cases} \qquad (4)$$

is the transformation kernel with $c(\theta) = \sqrt{\frac{1 - j \cot \theta}{2\pi}}$, $a(\theta) = \cot \theta / 2$, and $b(\theta) = \csc \theta$. The kernel $\mathcal{K}_\theta(t, \omega)$ is parameterized by an angle $\theta \in \mathbb{R}$ and p is an integer. For simplicity, we may write a, b, c instead of $a(\theta), b(\theta)$, and $c(\theta)$

$$F_{\phi}\{F_{\theta}[f]\}(\omega) = F_{\phi+\theta}[f](\omega).$$

Hence, the inverse fractional Fourier transform is given by

$$\{F_{\phi}\}^{-1} = F_{-\phi}.$$

For $f, g \in L^2(\mathbb{R})$, we have $\hat{f}_\theta, \hat{g}_\theta \in L^2(\mathbb{R})$,

$$\langle f, g \rangle = \langle \hat{f}_\theta, \hat{g}_\theta \rangle,$$

and hence,

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How about the convolution structure for the FRFT?

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Shift-Invariant Spaces

Shift-Invariant and Sampling Spaces Associated with the Fractional Fourier Transform, Ayush Bhandari and Ahmed Zayed, **IEEE Trans. Signal Processing, Vol. 60, April (2012)**

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Shift-invariant spaces generated by $\varphi \in L^2$

$$\mathcal{V}(\varphi) = \left\{ f : \quad f(t) = \sum_{n=-\infty}^{+\infty} c_k \varphi(t-k), \{c_k\} \in \ell_2 \right\}.$$

We require that $\{\varphi(t-n)\}_{n=-\infty}^{\infty}$ form a Riesz basis, hence

$$0 < \eta_1 \leqslant \sum_{n=-\infty}^{+\infty} |\widehat{\varphi}(\omega + 2\pi n)|^2 \leqslant \eta_2 < \infty.$$

Any Function in $V(\phi)$ can be viewed as a convolution of a sequences $\{c(k)\} \in \ell^2$ and a function $\phi \in L^2(\mathbb{R})$, where the convolution is defined as

$$(c(k) \star \phi)(t) = \sum_{k \in \mathbb{Z}} c(k)\phi(t-k).$$

Definition

A sampling space is a shift invariant space (SIS) in which the expansion coefficients are samples of the function, i.e.,

$$f(t) = \sum_k f(t_k)\psi(t - t_k)$$

where $\{t_k\}$ is a uniformly distributed sequence of real numbers.

An example of a SIS that is also a sampling space is the space of bandlimited functions to $[-\pi, \pi]$.

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Definition

A function $f \in L^2$ is said to be bandlimited to $[-\sigma, \sigma]$ if the support of its Fourier transform $\hat{f}$ is $[-\sigma, \sigma]$. We denote such a space by PW_σ.

It is known (Paley-Wiener Theorem) that $f \in PW_\sigma$ iff f is an entire function of exponential type σ that belongs to $L^2(\mathbb{R})$ when restricted to the real line

The Whittaker-Shannon-Kotel'nikov Sampling Theorem states that

Theorem

If $f \in PW_{\pi}$, then

$$f(t) = \sum_{k} f(k) \frac{\sin \pi (t-k)}{\pi (t-k)} = \sum_{k} f(k) \phi(t-k),$$

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where

$$\phi(t) = Sinc(t) = \frac{\sin \pi t}{\pi t}.$$

That is PW_π is a sampling space generated by the Sinc function.

The Zak Transform

The Zak transform of f is defined as

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It is a unitary transformation from $L^2(\mathbb{R})$ onto $L^2(Q)$, where Q is the unit square. It is also related to sampling spaces.

For example, if f belongs to the sampling space generated by ψ, then

$$\int_0^1 \frac{\sum_{k\in\mathbb{R}} \left|\hat{f}(\omega+k)\right|}{|Z_f(0,\omega)|} dw = \int_{\mathbb{R}} \left|\hat{\psi}(\omega)\right| d\omega < \infty, \quad (ZT) \qquad (5)$$

Going back to the fractional Fourier domain, we let $\lambda_\theta(t) = \exp\left(j\left(t^2/2\right)\cot\theta\right)$ be a modulation function.

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$$\text{Modulation/up-chirping: } \overrightarrow{x}(t) = x(t)\lambda_\theta(t)$$

$$\text{Demodulation/down-chirping: } \overleftarrow{x}(t) = x(t)\lambda_\theta^*(t).$$

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The *fractional convolution* of two input signals, $x(t)$ and $y(t)$ is defined as (A. Zayed, IEEE Sign. Proc. Letters, Vol. 5 (1998))

$$x(t) *_\theta y(t) = \sqrt{\frac{1 - j\cot\theta}{2\pi}} \lambda_\theta^*(t) \cdot \underbrace{([x(t)\lambda_\theta(t)] * [y(t)\lambda_\theta(t)])}_{\text{convolution of modulated inputs}}$$

$$= c\,(\theta)\,\lambda_\theta^*(t) \left\{ \vec{x}(t) * \vec{y}(t) \right\}, \tag{6}$$

where $c(\theta) = \sqrt{(1 - j\cot\theta)/2\pi}$.
Which leads to FrFT $\{x(t) *_\theta y(t)\} = \lambda_\theta^*(\omega) \cdot \widehat{x}_\theta(\omega)\widehat{y}_\theta(\omega)$.

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Let $t_k = k\Delta$, where $\Delta = 2\pi \sin\theta$, be a sequence of uniformly spaced real numbers.

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Define the discrete Fractional Fourier transform of a sequence $\{x(k)\}$

$$\hat{X}_\theta(w) = \sum_{k=-\infty}^{\infty} x(k)\, K_\theta(k, w) \tag{7}$$

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and define the convolution of a sequence and a function $\{x(k)\}$ with $\phi \in L^2(R)$ as

$$h(t) = \left(x(k) *'_\theta \phi\right)(t) = c\,(\theta)\,\overline{\lambda}_\theta(t) \sum_{k=-\infty}^{\infty} \vec{x}(k)\vec{\phi}(t - t_k)$$

Consider

$$\tilde{h}(t) = \lambda_\theta(t) h(t) = c(\theta) \sum_{k=\infty}^{\infty} \vec{x}(k) \vec{\phi}(t - t_k)$$

where $\vec{x}(k) = x(k)^{jat_k^2}, \vec{\phi}(t) = \phi(t) e^{jat^2}$, and $a = \cot\theta$.
The fractional spectrum of $h(t)$ is $\widehat{h}_\theta(\omega) = \lambda^*_\theta(\omega) \widehat{P}_\theta(\omega) \widehat{\phi}_\theta(\omega)$,
where $\widehat{P}_\theta$ is the discrete time fractional Fourier transform (DTFrFT) of the sequence $\left\{ p(n) = e^{jat_n^2} x(n) \right\}$.

Let $\{x(n)\} \in \ell_2$, $\phi \in L^2(R)$ and set

$$\psi(t) = e^{jat^2}\phi(t), \; p(n) = e^{jat_n^2}x(n)$$

and consider the chirp-modulated shift-invariant subspaces of L^2

$$V(\psi) = \text{cl}\left\{\tilde{f} \in L^2 : \tilde{f}(t) = c(\theta)\sum_{k=-\infty}^{\infty} p(k)\psi(t - t_k)\right\}$$

and

$$V(\phi) = \text{cl}\left\{f \in L^2 : f(t) = (x(n) *_\theta' \phi)(t) = \lambda_\theta^*(t)\tilde{f}(t), \; \tilde{f} \in V(\psi)\right\}$$

Then $\{\psi(t - t_k)\}$ is a Riesz basis for $V(\psi)$ if and only if there exist two positive constants $\eta_1, \eta_2 > 0$ such that

$$\eta_1 \leq \sum \left| \hat{\phi}_\theta(w + t_k) \right|^2 \leq \eta_2$$

for all $w \in [0, \Delta]$.
Other properties of shift-invariant spaces, such as sampling subspaces, etc, have been obtained in the fractional Fourier domain.

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We extend the Zak transform and some of its properties to the fractional Fourier transform domain.

Definition

We define the fractional Zak transform with angle θ of a function $f \in L^2(\mathbb{R})$ as

$$Z_{f,\theta}(t,w) = \frac{1}{\sqrt{\Delta}} \sum_{k\in\mathbb{Z}} f(t+k) e^{ja(\theta)\left(w^2+k^2\right)-jb(\theta)kw}.$$

$$\int_0^1 \frac{\sum_{k\in\mathbb{Z}} |\hat{f}_\theta(w+t_k)|}{|Z_{f,\theta}(0,w)|} dw = \int_{\mathbb{R}} |\hat{\psi}_\theta(w)| dw < \infty, \tag{8}$$

which is the analog of (ZT)

Theorem (PSF–FrFT)

Let $\tilde{f}(t)$ be a Δ–periodic function,

$$f\tilde{f}(t) = \sum_{k=-\infty}^{+\infty} f(t+k\Delta)e^{ja(\theta)(t+k\Delta)^2}$$
$$= \sum_{k=-\infty}^{+\infty} f(t+t_k)e^{ja(\theta)(t+t_k)^2},$$

where $t_k = k\Delta$ and $f(t)$ is assumed to be integrable. Then, the function $\tilde{f}(t)$ can be represented in FrFT domain as,

$$\sum_{k=-\infty}^{+\infty} f(t+t_k)e^{ja(\theta)(t+t_k)^2} = \frac{1}{c(\theta)\Delta}\sum_{n=-\infty}^{+\infty} e^{jnbt-ja(\theta)n^2}\hat{f}_\theta(n). \quad (9)$$

The Reproducing Kernel

The reproducing kernel for the space of bandlimited functions in the fractional Fourier transform domain is

$$K(x,t) = e^{ja(\theta)(x^2-t^2)} \frac{\sin\left[b(\theta)\sigma(x-t)\right]}{\pi(x-t)}, \tag{10}$$

which reduces to the reproducing kernel for the space of functions bandlimited to $[-\sigma,\sigma]$ when $\theta = \pi/2$;

The Sampling Theorem

We have the following sampling formula for functions that are bandlimited in the fractional Fourier transform domain

$$F(t)e^{ja(\theta)t^2} = 2\sum_{k=-\infty}^{+\infty} F(t_k)e^{ja(\theta)t_k^2}\frac{\sin\left[b(\theta)\sigma(t-t_k)\right]}{b(\theta)(t-t_k)}.$$

What about the energy concentration problem for the FRFT?

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This is the subject of the next lecture !

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Thank you for listening und Auf Wiedersehen