Sampling and the Energy Concentration Problem: A New Perspective

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New Trends and Directions in Harmonic Analysis, Fractional Operator Theory, and Image Analysis, Inzell, Germany, September 17 - 21, 2012

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The Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem plays an important role in communication engineering because it enables engineers to reconstruct analog signals from their samples at a discrete set of points; hence, converting analog signals into digital signals and back.

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$F \in L^2(R)$.
The space PW_σ is the same as the Bernstein space B_σ^2.

The Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem

The WSK states that if $f \in PW_\sigma$, then it can be reconstructed from its samples, $f(k\pi/\sigma)$. The construction formula is

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin(\sigma t - k\pi)}{(\sigma t - k\pi)}, \\ &= \sum_{k=-\infty}^{\infty} f(t_k) \operatorname{sinc}(\sigma\pi)(t - t_k) \qquad t \in \mathbb{R}, \end{aligned}$$

where $t_k = k\pi/\sigma$ and sinc $x = \sin \pi x/\pi x$.

It is known that the space B_σ^2 is a reproducing-kernel Hilbert space with reproducing kernel, $\sin \sigma t/\pi t$, i.e., for any $f \in B_\sigma^2$ we have

$$\int_{-\infty}^{\infty} f(x) \frac{\sin \sigma(t-x)}{\pi(t-x)}\, dx = f(t). \tag{2}$$

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- The WSK theorem has been generalized in many directions. One of the earliest generalizations was due to Parzen , who extended it to $N-$dimensions by giving a sampling formula for signals that are bandlimited to the $N-$dimensional cube $[-\pi, \pi]^N$.
- He showed that if f is bandlimited to the $N-$dimensional cube $[-\pi, \pi]^N$ then

$$f(t) = \sum_{n_1,\cdots,n_N=-\infty}^{\infty} f(n) \prod_{i=1}^{N} \frac{sin(\pi(t_i - n_i))}{\pi(t_i - n_i)}, \qquad (3)$$

where $n = (n_1, \cdots, n_N)$, and $t = (t_1, \cdots, t_N)$ is a multi-index.

In general, sampling in several variables is more difficult to obtain, especially for functions that are bandlimited to general domains in $\mathbb{R}^N$ because the sampling functions depend on the geometry of the domain.

The energy of a signal is defined as $E = \int_{-\infty}^{\infty} |f(t)|^2 dt$. An important question in engineering is that given $T > 0$, find a bandlimited function with maximum energy concentration in $(-T, T)$. Equivalently, let us maximize

$$\alpha^2 = \frac{\int_{-T}^{T} |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt} \leq 1.$$

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If f has support in $(-T, T)$, then $\alpha = 1$, and we are done. But this cannot happen since f is an entire function.

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- The prolate spheroidal wave functions, which were first discovered by Niven, *Philosophical Trans. Roy. Soc. London,* (1880), are a special case of the spheroidal wave functions.

The PSWF are bounded eigenfunctions of the differential equation

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dw}{dx}\right)+\left(\lambda+\gamma^2(1-x^2)\right)w=0 \tag{4}$$

For $\gamma = 0$, they are reduced to the Legendre functions.

Let $\gamma = \tau\sigma$. The prolate spheroidal wave functions, $\varphi_{n,\sigma,\tau}$, are eigenfunctions of the differential operator

$$(\tau^2 - t^2)\frac{d^2\varphi_{n,\sigma,\tau}}{dt^2} - 2t\frac{d\varphi_{n,\sigma,\tau}}{dt} - \sigma^2 t^2 \varphi_{n,\sigma,\tau} = \nu_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}, \quad (5)$$

where $\nu_{n,\sigma,\tau}$ are the eigenvalues.

Double Orthogonality Relations

It is known that $\varphi_{n,\sigma,\tau}$ satisfy a number of interesting relations, chief among them are the dual orthogonality relations

$$\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(t)\varphi_{m,\sigma,\tau}(t)dt = a_n\delta_{m,n} \tag{6}$$

and

$$\int_{-\infty}^{\infty} \varphi_{n,\sigma,\tau}(t)\varphi_{m,\sigma,\tau}(t)dt = \delta_{m,n}. \tag{7}$$

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- They also form an orthogonal basis for $L^2(-\tau, \tau)$ and orthonormal basis for a subspace of $L^2(\mathbb{R})$, namely, PW_σ.
- Note that the double orthogonality of the PSWFs distinguishes them from other classical orthogonal bases of $L^2([-1, 1])$, such as the Legendre or the Jacobi polynomials.

Integral Equations

$$\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(x) \frac{\sin \sigma(t-x)}{\pi(t-x)} dx = \lambda_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}(t), \tag{8}$$

where $\lambda_n = \lambda_{n,\tau,\sigma}$ are the eigenvalues, and the second is

$$\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(t) e^{i\sigma wt/\tau} dt = \gamma_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}(w). \tag{9}$$

Because they are bandlimited to $(-\sigma, \sigma)$, we have from (2)

$$\int_{-\infty}^{\infty} \varphi_{n,\sigma,\tau}(x) \frac{\sin \sigma(t-x)}{\pi(t-x)} dx = \varphi_{n,\sigma,\tau}(t), \tag{10}$$

We have

$$\frac{\sin \sigma(t-x)}{\pi(t-x)} = \sum_{n=0}^{\infty} \varphi_{n,\sigma,\tau}(t)\varphi_{n,\sigma,\tau}(x) \tag{11}$$

and the discrete orthogonality relations

$$\sum_{n=0}^{\infty} \varphi_{n,\sigma,\tau}(k)\varphi_{n,\sigma,\tau}(m) = \delta_{k,m}. \tag{12}$$

Fourier Transform

Furthermore, their Fourier transforms satisfy the relations

$$\int_{-\infty}^{\infty} e^{-itw}\varphi_{n,\sigma,\tau}(t)dt = (-i)^n\sqrt{\frac{2\pi\tau}{\sigma\lambda_n}}\varphi_{n,\sigma,\tau}(\tau w/\sigma)\chi_\sigma(w) \qquad (13)$$

and

$$\int_{-\tau}^{\tau} e^{-itw}\varphi_{n,\sigma,\tau}(t)dt = (-1)^n\sqrt{\frac{2\pi\tau\lambda_n}{\sigma}}\varphi_{n,\sigma,\tau}(\pi w/\sigma) \qquad (14)$$

where $\chi_\sigma(w)$ is the characteristic function of $(-\sigma, \sigma)$.

Sampling with the PSWF

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Sampling with the PSWF

G. Walter and X. Shen have recently shown (STSIP, 2003) that the PSWFs, can replace the *Sinc* function in the sampling theorem. That is any $\pi-$bandlimited function f,

$$f(t) = \sum_{n=0}^{\infty} \left(\sum_{k=-\infty}^{\infty} \varphi_n(k) f(k) \right) \varphi_n(t) = \sum_{n=0}^{\infty} \tilde{f}(n) \varphi_n(t) \qquad (15)$$

where $\tilde{f}(n) = \left(\sum_{k=-\infty}^{\infty} \varphi_n(k) f(k) \right)$, and $\{\varphi_n, n \in \mathbb{N}\}$ is a sequence of $\pi-$bandlimited PSWFs with a concentration interval $(-1, 1)$.

An advantage of using the prolate spheroidal wave functions over the sinc function is that the prolate spheroidal wave functions are more concentrated on finite intervals than the sinc function and have much faster decay at infinity than the sinc function.

One of the goals of this talk is to extend Walter and Shen's result to N dimensions.

One of the goals of this talk is to extend Walter and Shen's result to N dimensions. Although this can be done in a direct way, we will obtain it as a special case of a more general result that we shall outline at the end of the talk.

Recall that

$$\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(x) \frac{\sin \sigma(t-x)}{\pi(t-x)} dx = \lambda_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}(t), \qquad (16)$$

where $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots$.

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Recall that

$$\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(x) \frac{\sin \sigma(t-x)}{\pi(t-x)} dx = \lambda_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}(t), \tag{16}$$

where $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots$. The above equation can be extended to all $t \in \mathbb{R}$.

Among all functions bandlimited to $(-\sigma, \sigma)$ that are concentrated in $(-\tau, \tau)$ the most efficient are the PSWF $\phi_{n,\sigma,\tau}(t)$.

Among all functions bandlimited to $(-\sigma, \sigma)$ that are concentrated in $(-\tau, \tau)$ the most efficient are the PSWF $\phi_{n,\sigma,\tau}(t)$. Recall that the concentration is measured by

$$\alpha^2(\tau) = \frac{\int_{-\tau}^{\tau} |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}.$$

Among all functions bandlimited to $(-\sigma, \sigma)$ that are concentrated in $(-\tau, \tau)$ the most efficient are the PSWF $\phi_{n,\sigma,\tau}(t)$. Recall that the concentration is measured by

$$\alpha^2(\tau) = \frac{\int_{-\tau}^{\tau} |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}.$$

It can be shown that $\alpha^2 = \lambda$; the eigenvalue parameter of the integral equation (16). Hence, the largest α is attained by $\alpha = \lambda_0$.

That is the maximum concentration is achieved by $\phi_{0,\sigma,\tau}(t)$, for which $\alpha^2 = \lambda_0$. To simplify the notation, we normalize the prolate spheroidal wave functions so that they are orthonormal on $(-1, 1)$, i.e., we set $\tau = 1$ or $\gamma = \sigma$ and denote $\varphi_{n,\sigma,1}$ by φ_n.

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The fact that they form orthogonal bases for two different spaces on a finite and an infinite interval and solve the concentration problem appears to be unique.

Are there other systems possessing this property?

Answering this question is another goal of this talk. We will show that the answer is affirmative and we will give another example.

Let E be an arbitrary set and $\mathcal{F}(E)$ be a linear space composed of all complex valued functions defined on E. Let $\mathcal{H}$ be a Hilbert space with linear product $\langle\, , \,\rangle_{\mathcal{H}}$. Let $h : E \to \mathcal{H}$ be a vector-valued function from E into $\mathcal{H}$. Consider the linear mapping L from $\mathcal{H}$ into $\mathcal{F}(E)$ defined by

$$f(p) = (LF)(p) = \langle F, h(p) \rangle_{\mathcal{H}}\,, \tag{17}$$

where $LF = f,\ F \in \mathcal{H},\ f \in \mathcal{F}(E)$.

Let $\tilde{\mathcal{H}}$ denote the range of L and $N(L)$ be the null space of L. Let $M = \mathcal{H} \ominus N(L)$, and denote by P_M the orthogonal projection from $\mathcal{H}$ into M. Saitho (Proc. AMS, 1983) has shown that $(\tilde{\mathcal{H}}, \langle\, , \,\rangle_{\tilde{\mathcal{H}}})$ is a reproducing-kernelHilbert space that is isometric to $(M, \langle\, , \,\rangle_{\mathcal{H}})$, where

$$\langle f, g\rangle_{\tilde{\mathcal{H}}} = \langle LF\,,\, LG\rangle_{\tilde{\mathcal{H}}} = \langle P_M F\,,\, P_M G\rangle_{\mathcal{H}}\,. \tag{18}$$

Now we apply these general results to a specific case.

Now we apply these general results to a specific case. Let $d\mu$ be a σ-finite positive measure and T be a $d\mu$ -measurable set. Consider the Hilbert Space $\mathcal{H} = L^2(T, d\mu)$ consisting of all complex valued functions F such that

$$\|F\|^2_{L^2(T,d\mu)} = \int_T |F(t)|^2 d\mu(t) < \infty.$$

Let E be an arbitrary set and $h(t, p)$ be a fixed complex-valued function on $T \times E$, such that

$$h(t, p) \in L^2(T, d\mu) \text{ for any } p \in E.$$

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Let L be the linear mapping $L : L^2(T, d\mu) \to \mathcal{F}(E)$ defined by

$$f(p) = (LF)(p) = \int_T F(t)\bar{h}(t,p)d\mu(t), \quad F \in L^2(T, d\mu). \quad (19)$$

Then, the set of all such f's is a reproducing-kernel Hilbert space $\tilde{\mathcal{H}} = \mathcal{H}_K$ with reproducing kernel $K(p, q)$ with $f(q) =< f, K(.\,, q) >_{\tilde{\mathcal{H}}}$.

It is not difficult to see that the the function

$$K(p,q)=\int_{\mathcal{T}} h(t,q)\overline{h}(t,p)d\mu(t), \tag{20}$$

is positive definite on $\mathcal{E}$, i.e.,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}K(p_i,p_j)\geq 0,$$

for any finite set $\{p_i\}$ of $\mathcal{E}$.

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Then it follows that $K(p, q)$ is a reproducing kernel for some Hilbert space of functions defined on $\mathcal{E}$. In fact, the set of all f's given by (17), i.e., the range of the operator L, is a reproducing-kernel Hilbert space $\widetilde{\mathcal{H}}$ whose reproducing kernel is given by (20) so that $f(q) = < f, K(., q) >_{\widetilde{\mathcal{H}}}$; see Saitoh, Proc. Amer. Math. Soc., Vol. 89 (1983).

Moreover, The reproducing kernel is

$$K(p,q) = \langle h(p), h(q) \rangle_{\mathcal{H}} \,. \tag{21}$$

and

$$\|f\|_{\tilde{\mathcal{H}}} = \|P_M F\|_{\mathcal{H}} = \|F\|_{\mathcal{H}} \,. \tag{22}$$

Theorem

Let $\mathcal{T}$ be a compact, connected subset of $\mathbb{R}^N$, and $\mathcal{E}$ be an open, connected subset of $\mathbb{R}^N$ containing $\mathcal{T}$, i.e, $\mathcal{T} \subset \mathcal{E} \subset \mathbb{R}^N$.

Theorem

Let $\mathcal{T}$ be a compact, connected subset of $\mathbb{R}^N$, and $\mathcal{E}$ be an open, connected subset of $\mathbb{R}^N$ containing $\mathcal{T}$, i.e, $\mathcal{T} \subset \mathcal{E} \subset \mathbb{R}^N$. Let $h(t, p)$ be complex-valued, symmetric, and continuous on $\mathcal{T} \times \mathcal{E}$, and assume that $\{h(t, p)\}_{p \in \mathcal{E}}$ is complete in $\mathcal{H} = L^2(\mathcal{T}, d\mu)$.

i) Then there exist a reproducing kernel Hilbert space, $\widetilde{H}$, comprising of functions defined on $\mathcal{E}$ and an orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$ of $L^2(\mathcal{T}, d\mu)$ with the property that ϕ_n can be naturally extended to functions Φ_n defined on $\mathcal{E}$ such that $\Phi_n \in \widetilde{H}$ and

ii) $< \Phi_m, \Phi_n >_{\widetilde{H}} = \delta_{m,n}/\mu_n$, with $\mu_n = |\lambda_n|^2$, where λ_n are the eigenvalues of a compact operator on $L^2(\mathcal{T}, d\mu)$; namely L.

ii) $< \Phi_m, \Phi_n >_{\widetilde{H}} = \delta_{m,n}/\mu_n$, with $\mu_n = |\lambda_n|^2$, where λ_n are the eigenvalues of a compact operator on $L^2(\mathcal{T}, d\mu)$; namely L.

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$$L(F)(p) = \int_{\mathcal{T}} F(t)\overline{h}(t,p)d\mu(t).$$

iii) $\{\phi_n\}_{n\in\mathbb{N}}$ are solutions of the Fredholm integral equation of the second kind

$$\mathbf{K}[\phi_n] = \int_{\mathcal{T}} \phi_n(x)K(x,p)d\mu(x) = \mu_n\phi_n(p), \quad p \in \mathcal{T}, \qquad (23)$$

where $K(x,p)$ is the reproducing kernel of $\widetilde{H}$.

iv)

$$K(p,q) = \sum_{n=0}^{\infty} \mu_n \phi_n(p)\phi_n(q). \tag{24}$$

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v)

$$f(p) = \sum_{n\in\mathbb{N}} f(p_n)K(p,p_n) \tag{25}$$

and the functions $\{\phi_n\}$ satisfy the discrete orthogonality relation

$$\sum_{n\in\mathbb{N}} \mu_n \phi_n(p_k)\phi_n(p_m) = \delta_{k,m}. \tag{26}$$

vi) For any function f of the form

$$f(p) = \int_{\mathcal{T}} F(t)\overline{h}(t,p)d\mu(t), \quad F \in L^2(\mathcal{T}, d\mu), \tag{27}$$

we have

$$f(t) = \sum_{n\in\mathbb{N}} \mu_n \left(\sum_{k\in\mathbb{N}} \phi_n(p_k) f(p_k) \right) \phi_n(t). \tag{28}$$

This is a generalization of Walter and Shen's sampling formula

Example 1

Example 1:
Let $h(t,p) = e^{-i\langle t,p\rangle}$, $d\mu(t) = (2\pi)^{-n/2}dt$, and consider the finite Fourier integral operator

$$LF(p) = \int_{S_c} F(t)e^{i\langle t,p\rangle}d\mu(t), \quad F \in L^2(S_c),$$

where $S_c = [-c,c]^n$ is the n dimensional cube with side $2c$ centered at the origin.

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where $S_c = [-c,c]^n$ is the n dimensional cube with side $2c$ centered at the origin.
The adjoint operator is

$$L^*F(t) = \int_{S_c} F(p)e^{-i\langle t,p\rangle}d\mu(p).$$

The operator L is compact and hence it has eigenfunctions and eigenvalues that are not necessarily real.

Furthermore, one can easily show that

$$\mathbf{K}F(q) = L\left(L^{*}F\right)(t) = L^{*}\left(LF\right)(t) = \int_{S_c} F(p)K(p,q)d\mu(p),$$

where

$$\begin{aligned} K(p,q) &= \int_{S_c} e^{i\langle t, q-p\rangle} d\mu(t) \\ &= \left(\frac{2}{\pi}\right)^{n/2} \prod_{k=0}^{n} \frac{\sin c(q_k - p_k)}{(q_k - p_k)}. \end{aligned}$$

The operator **K** is a compact, self-adjoint operator whose kernel function is the reproducing kernel of the reproducing-kernel Hilbert space of bandlimited functions to S_c, and in which

$$f(q) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(p)k(p,q)dp$$

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for any bandlimited function f.
The operator **K** has real eigenvalues and eigenfunctions which are the n-dimensional prolate spheroidal wave functions. The sampling formula given by our main theorem is the n-dimensional generalization of Walter and Shen's sampling formula.

Example 2

Let $D = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\right\}$ be the unit disc in $\mathbb{R}^2$ and consider the integral operator

$$LF(x) = \int_D e^{ic\langle x,y\rangle} F(y)dy = \int_D e^{ic(x_1y_1+x_2y_2)} F(y_1, y_2)dy_1 dy_2, \tag{29}$$

To find the eigenvalues and eigenvectors of L, we convert to polar coordinates to obtain

$$\alpha\psi(\rho, \phi) = \int_0^1 rdr \int_0^{2\pi} \psi(r, \theta)e^{icr\rho\cos(\phi-\theta)}d\theta.$$

If we write $\psi(r,\theta) = R(r)\Theta(\theta)$, we obtain

$$\begin{aligned}\alpha\psi(\rho,\phi) &= \sum_{k=-\infty}^{\infty} (i)^k e^{ik\phi} \int_0^1 R(r) J_k(c\rho r) r dr \int_0^{2\pi} \Theta(\theta) e^{-ik\theta} d\theta \\ &= \sum_{k=-\infty}^{\infty} (i)^k e^{ik\phi} A_k(\rho)(2\pi)\hat{\theta}_k, \qquad (30)\end{aligned}$$

where

$$\alpha\psi(\rho,\phi) = \sum_{k=-\infty}^{\infty} (i)^k e^{ik\phi} \int_0^1 J_k(c\rho r) r dr \int_0^{2\pi} \psi(r,\theta) e^{-ik\theta} d\theta.$$

which leads to finding the eigenfunctions and eigenvalues of the finite Hankel transform integral operator

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$$\mathcal{L}R(\rho) = \int_0^1 R(r)J_k(c\rho r)rdr,$$

The eigenfunctions are closely related to the Zernike polynomials which appear in aberration theory in optics

$$\phi_{k,n}(x) = \frac{1}{\gamma_{k,n}} \sum_{j=0}^{\infty} d_j^{k,n}(c) \frac{J_{k+2j+1}(cx)}{\binom{k+j}{j}\sqrt{cx}}, \quad 0 \le x.$$

To obtain the equivalent of Eq. (23), we note that

$$L^*F(x) = \int_D e^{-i\langle x,y\rangle} F(y)d\mu(y),$$

and therefore,

$$\mathbf{K}F(z) = \int_D F(y)K(y,z)d\mu(y),$$

where

$$K(y,z) = \int_D e^{i\langle x,z-y\rangle} d\mu(x);$$

hence,

$$f(z) = \int_{\mathbb{R}^2} f(y)K(y,z)dy,$$

for any Hankel bandlimited function f.

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By making a change of variables we obtain

$$\begin{aligned} K(y,z) &= \int_D e^{i\langle x, z-y\rangle} d\mu(x) \\ &= \frac{1}{2\pi}\int_0^1 r dr \int_0^{2\pi} e^{ir\rho\cos(\phi-\theta)} d\theta \\ &= \frac{J_1(\rho)}{\rho} = \frac{J_1\left(\sqrt{(z_1-y_1)^2+(z_2-y_2)^2}\right)}{\sqrt{(z_1-y_1)^2+(z_2-y_2)^2}}, \end{aligned}$$

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where the last integral follows from the relation $\int x^{p+1} J_p(x) dx = x^{p+1} J_{p+1}(x)$. The eigenfunctions of **K** are the same as those of L. Therefore, they possess all the properties listed in the main Theorem.

Example 3 (The Finite Hankel Transform)

Let $h(r,\rho) = \sqrt{r\rho}J_k(r\rho)$ and consider the finite Hankel transform

$$LF(\rho) = \int_0^1 F(r)J_k(\rho r)\sqrt{r\rho}dr,$$

which is self-adjoint and compact. It is easy to see that

$$\mathbf{K}F(\rho) = \int_0^1 F(r)K(r,\rho)dr, \quad \text{where}$$

$$\begin{aligned} K(r,\rho) &= \sqrt{r\rho}\int_0^1 J_k(r\eta)J_k(\rho\eta)\eta d\eta \\ &= \frac{\sqrt{r\rho}}{\rho^2 - r^2}\left[\rho J_{k+1}(\rho)J_k(r) - rJ_{k+1}(r)J_k(\rho)\right] \end{aligned}$$

is the reproducing kernel of the reproducing-kernel Hilbert

Thus, $f \in \mathcal{H}$, we have

$$f(\rho) = \int_0^\infty f(r)K(\rho, r)dr.$$

The eigenfunctions $\{\phi_{k,n}\}$ and the eigenvalues of L and $\mathbf{K}$ are given as in the previous example.

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$$f(\rho) = \int_0^\infty f(r)K(\rho, r)dr.$$

The eigenfunctions $\{\phi_{k,n}\}$ and the eigenvalues of L and **K** are given as in the previous example. According to the Main Theorem, the eigenfunctions satisfy the conclusions of the Theorem including the analogue of Walter-Shen sampling formula.

Thank you for listening; any questions?