Construction of smooth windows generating dual pairs of Gabor - and wavelet frames

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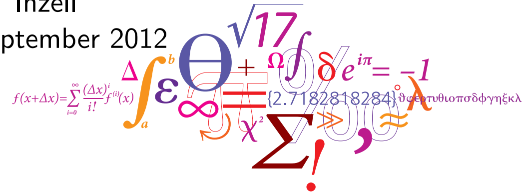
Inzell
20. September 2012



Gabor and wavelet frames in $L^2(\mathbb{R})$ - Fundamentals

Definition

Let $a, b \in \mathbb{R}$ and $c > 0$. We now define

$$(i)\ \text{Translation by } a,\ T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (T_a f)(x) = f(x-a),$$

$$(ii)\ \text{Modulation by } b,\ E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (E_b f)(x) = e^{2\pi i b x} f(x),$$

$$(iii)\ \text{Dilation by } c,\ D_c : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (D_c f)(x) = \frac{1}{\sqrt{c}} f\left(\frac{x}{c}\right).$$

- Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$. The collection of functions $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is referred to as the associated *Gabor system*.
- Let $\psi \in L^2(\mathbb{R})$ and $a > 1, b > 0$. The collection of functions $\{D_{a^j} T_{bk} \psi\}_{j,k \in \mathbb{Z}}$ is referred to as the associated *wavelet system*.

Gabor and wavelet frames in $L^2(\mathbb{R})$ - Fundamentals

Definition

A sequence $\{f_k\}$ of elements in a separable Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exists constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \ \forall \ f \in \mathcal{H}. \tag{1}$$

Frames allow a convenient way of obtaining series expansions of functions in $\mathcal{H}$:

Theorem

Assume that $\{f_k\}$ is a frame. Then there exists at least one other frame $\{g_k\}$, such that

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \ \forall \ f \in \mathcal{H}. \tag{2}$$

Contents

- New methods to construct dual pairs of Gabor - and wavelet frames.
- Relations to the *Daubechies polynomials*.
- The *Meyer scaling functions*.
- Construction of smooth windows generating pairs of dual Gabor - and wavelet frames.
- Construction of an explicitly given smooth window generating a dual frame for a class of wavelet frames.

A new construction of dual windows

Theorem

Let $K \in \mathbb{N}_0$ and let $b \in]0, \frac{1}{4K+2}]$. Let g be a real-valued bounded function with $\operatorname{supp} g \subseteq [-(2K+1), 2K+1]$, for which

$$\left| \sum_{n \in \mathbb{Z}} g(x+n) \right| \geq A, \; x \in [0,1], \tag{3}$$

for a constant $A > 0$. Define $\tilde{G}$ by

$$\tilde{G}(x) := \sum_{k \in \mathbb{Z}} g(x+2k), \; x \in [-1,1], \tag{4}$$

Take $\tilde{H} \in L^2(\mathbb{R})$ to satisfy,

$$\tilde{G}(x-1)\tilde{H}(x-1) + \tilde{G}(x)\tilde{H}(x) = 1, \; a.e. \; x \in [0,1]. \tag{5}$$

A new construction of dual windows

Define h by

$$h(x) = b \sum_{k=-K}^{K} \tilde{H}(x + 2k). \tag{6}$$

Then $\{E_{mb}T_n g\}_{n,m\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{n,m\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$, and h is supported in $[-(2K+1), 2K+1]$.

The connection to the Daubechies polynomials

We assume that g satisfies the conditions of Theorem 4. Furthermore we assume that

$$\sum_{k\in\mathbb{Z}} g(x+k) = 1, \; a.e. \; x \in \mathbb{R}.$$

Choose $\tilde{H}$ as to satisfy

$$\tilde{G}(x-1)\tilde{H}(x-1) + \tilde{H}(x)\tilde{G}(x) = 1, \; a.e. \; x \in [0,1]. \tag{7}$$

For any $N \in \mathbb{N}$ the *Daubechies polynomial* of degree $N-1$ is given by

$$P_{N-1}(x) = \sum_{k=0}^{N-1} \binom{2N-1}{k} x^k (1-x)^{N-1-k}, \; x \in \mathbb{R}. \tag{8}$$

For each $N \in \mathbb{N}$

$$(1-x)^N P_{N-1}(x) + x^N P_{N-1}(1-x) = 1, \; x \in \mathbb{R}. \tag{9}$$

The connection to the Daubechies polynomials

Lemma

Let g be a real-valued bounded function with supp $g \subseteq [-(2K+1), 2K+1]$, $K \in \mathbb{N}_0$ *such that*

$$\sum_{k \in \mathbb{Z}} g(x+k) = 1, \ a.e. \ x \in [0,1].$$

Define $\tilde{G}$ *by (4). For any* $N \in \mathbb{N}$ *define*

$$P_{2N-2}(x) = \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-x)^{2(N-1)-k} x^k, \ N \in \mathbb{N}. \tag{10}$$

We then have

$$\tilde{G}(x) P_{2N-2}(\tilde{G}(x-1)) + \tilde{G}(x-1) P_{2N-2}(\tilde{G}(x)) = 1, \ a.e. \ x \in [0,1].$$

Windows generating dual pairs of Gabor frames

Theorem

Let g be a real-valued bounded function with $\operatorname{supp} g \subseteq [-(2K+1), 2K+1]$, $K \in \mathbb{N}_0$ such that

$$\sum_{k\in\mathbb{Z}} g(x+k) = 1, \text{ a.e. } x \in [0,1].$$

Define the function $\tilde{G}$ by (4). Let $N \in \mathbb{N}$ and define P_{2N-2} by (10). Now let

$$\tilde{H}(x) = \begin{cases} P_{2N-2}(\tilde{G}(x+1)), & x \in [-1,0[\\ P_{2N-2}(\tilde{G}(x-1)), & x \in [0,1] \\ 0 & x \notin [-1,1]. \end{cases} \tag{11}$$

Let $b \in]0, \frac{1}{4K+2}]$ and define h by (6). Then the functions g and h will generate dual frames $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Windows generating dual pairs of wavelet frames

Theorem

Let $n \in \mathbb{N}$, $a > 1$ and $\psi \in L^2(\mathbb{R})$. Let $\widehat{\psi}$ be a real-valued bounded function with supp $\widehat{\psi} \subset [-a^{c+2n+1}, -a^{c-(2n+1)}] \cup [a^{c-(2n+1)}, a^{c+2n+1}]$ for some $c \in \mathbb{Z}$. Assume that

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(a^j \xi) = 1, \ a.e. \ \xi \in \mathbb{R}, \tag{12}$$

and define Ψ by

$$\Psi(\xi) := \sum_{j=-n}^{n} \widehat{\psi}(a^{2j}\xi), \ \xi \in [-a^{c+1}, -a^{c-1}] \cup [a^{c-1}, a^{c+1}]. \tag{13}$$

Let $N \in \mathbb{N}$ and define

$$P_{2N-2}(x) = \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-x)^{2(N-1)-k} x^k, \ x \in \mathbb{R}.$$

Windows generating dual pairs of wavelet frames

Furthermore define the function $\widehat{\tilde{\Psi}}$ by

$$\widehat{\tilde{\Psi}}(\xi) = \begin{cases} P_{2N-2}(\Psi(a^{-1}\xi)), & \xi \in [-a^{c+1}, -a^c[\cup [a^c, a^{c+1}] \\ \\ P_{2N-2}(\Psi(a\xi)), & \xi \in [-a^c, -a^{c-1}] \cup [a^{c-1}, a^c[. \end{cases} \tag{14}$$

Let $b \in]0, 2^{-1}a^{-(c+2n+1)}]$ and define $\tilde{\psi}$ by

$$\tilde{\psi}(x) = b \sum_{j=-n}^{n} a^{-2j} \tilde{\Psi}(a^{-2j}x), \; x \in \mathbb{R}. \tag{15}$$

Then the functions ψ and $\tilde{\psi}$ generate dual frames $\{D_{a^j} T_{bk} \psi\}_{j,k \in \mathbb{Z}}$ and $\{D_{a^j} T_{bk} \tilde{\psi}\}_{j,k \in \mathbb{Z}}$ for $L^2(\mathbb{R})$.

New explicit constructions - Desired properties

For any $k \in \mathbb{N}$ we seek to construct g, h as to fulfill:

- $g, h \in C^k(\mathbb{R})$.
- $\operatorname{supp} g \subseteq [-1, 1],\ \operatorname{supp} h \subseteq [-1, 1]$.
- $g(x) = g(-x),\ h(x) = h(-x), \forall x \in \mathbb{R}$.

In turn the pair of dual generators will have important features including:

- By choosing $k \in \mathbb{N}$ sufficiently large, polynomial decay of $\widehat{g}$ and $\widehat{h}$ of any desired order can be obtaind.
- The largest possible range of the modulation parameter is accessible ($b \in]0, 1/2]$).
- Compact support of g, h insures perfect time-localization, whereas symmetry reduces computational effort.

New explicit constructions - The Meyer scaling functions

- Used by Y.Meyer to construct the first examples of smooth wavelets.
- Existence of an accessible construction algorithm.
- Important properties

  For any $k \in \mathbb{N} \cup \{\infty\}$ the associated Meyer scaling function $\tilde{b}$ satisfies

  - $\tilde{b} \in C^k(\mathbb{R})$,
  - $\tilde{b}(x) = 1, \ |x| \leq 1/3$,
  - $\tilde{b}(x) = 0, \ |x| \leq 2/3$,
  - $\sum_{k \in \mathbb{Z}} |\tilde{b}(x+k)|^2 = 1, \ a.e. \ x \in \mathbb{R}.$

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New explicit constructions

Theorem

Let $k \in \mathbb{N} \cup \{\infty\}$. Let $\tilde{b}$ be the associated Meyer scaling function. For any $N \in \mathbb{N},\ N > 1$ define P_{2N-2} by (10). Define $\tilde{H}$ by

$$\tilde{H}(x) = \begin{cases} P_{2N-2}(\tilde{b}(x+1)^2), & x \in [-1, 0[\\ P_{2N-2}(\tilde{b}(x-1)^2), & x \in [0, 1] \\ 0, & x \notin [-1, 1]. \end{cases} \tag{16}$$

Let $b \in]0, 1/2]$ and define

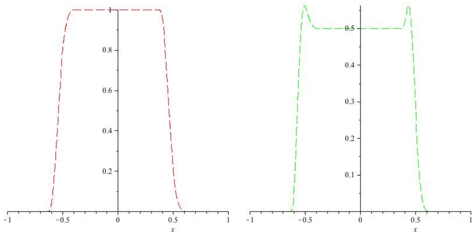
$$h(x) = b\tilde{H}(x),\ x \in \mathbb{R}. \tag{17}$$

Then the functions $\tilde{b}^2$ and h generate dual frames $\{E_{mb}T_n\tilde{b}^2\}_{n,m\in\mathbb{Z}}$ and $\{E_{mb}T_nh\}_{n,m\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

New explicit constructions - Important properties

Let $k \in \mathbb{N} \cup \{\infty\}$. Let $b \in]0, 1/2]$. Then h will have the following properties

- $\operatorname{supp} h \subseteq [-2/3, 2/3]$,
- $h(x) = b, \ \forall\, x \in [-1/3, 1/3]$,
- $h \in C^k(\mathbb{R})$,
- $h(x) = h(-x), \ \forall\, x \in \mathbb{R}$.



Figur: Plots of (a) The C^∞-*Meyer scaling function* $\tilde{b}$, (b) The associated C^∞-dual generator h, for $N = 2, \ b = 1/2$.

Windows generating dual pairs of wavelet frames

Theorem

Let $n \in \mathbb{N}$, $a > 1$ and $\psi \in L^2(\mathbb{R})$. Let $\widehat{\psi}$ be a real-valued bounded function with supp $\widehat{\psi} \subset [-a^{c+2n+1}, -a^{c-(2n+1)}] \cup [a^{c-(2n+1)}, a^{c+2n+1}]$ for some $c \in \mathbb{Z}$. Assume that $\widehat{\psi}$ satisfies the partition of unity condition and define Ψ by (13). Let $N \in \mathbb{N}$ and define

$$P_{2N-2}(x) = \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-x)^{2(N-1)-k} x^k, \; x \in \mathbb{R}.$$

Furthermore define the function $\widehat{\tilde{\Psi}}$ by

$$\widehat{\tilde{\Psi}}(\xi) = \begin{cases} P_{2N-2}(\Psi(a^{-1}\xi)), & \xi \in [-a^{c+1}, -a^c[\cup [a^c, a^{c+1}] \\ P_{2N-2}(\Psi(a\xi)), & \xi \in [-a^c, -a^{c-1}] \cup [a^{c-1}, a^c[. \end{cases} \tag{18}$$

Let $b \in]0, 2^{-1}a^{-(c+2n+1)}]$ and define $\tilde{\psi}$ by

$$\tilde{\psi}(x) = b \sum_{j=-n}^{n} a^{-2j} \tilde{\Psi}(a^{-2j}x), \; x \in \mathbb{R}. \tag{19}$$

Then the functions ψ and $\tilde{\psi}$ generate dual pairs of wavelet frames for $L^2(\mathbb{R})$.

New explicit constructions - Joint dual window

Let $N = 1$. It then follows from Theorem 9 that

$$\widehat{\tilde{\Psi}}(\xi) = \chi_{[-a^{c+1}, -a^{c-1}] \cup [a^{c-1}, a^{c+1}]}(\xi), \ \xi \in \mathbb{R}.$$

Since $\widehat{\tilde{\Psi}}(\xi) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ it follows by the *Inversion theorem* that

$$\begin{aligned}
\tilde{\Psi}(x) &= \int_{a^{c-1}}^{a^{c+1}} e^{-2\pi i x \xi} d\xi + \int_{a^{c-1}}^{a^{c+1}} e^{2\pi i x \xi} d\xi \\
&= 2 \int_{a^{c-1}}^{a^{c+1}} \cos(2\pi x \xi) d\xi \\
&= \frac{\sin(2\pi x a^{c+1}) - \sin(2\pi x a^{c-1})}{\pi x} \\
&= \frac{2 \cos\left(a^c \left(a + \frac{1}{a}\right) \pi x\right) \sin\left(a^c \left(a - \frac{1}{a}\right) \pi x\right)}{\pi x}, \ x \neq 0.
\end{aligned}$$

New explicit constructions - Joint dual window

Theorem

Let ψ satisfy the conditions of Theorem 9. For any fixed choice of the parameters $a > 1, n \in \mathbb{N}, c \in \mathbb{Z}$ and $b \in]0, 2^{-1}a^{-(c+2n+1)}]$, the function ψ has a dual window given by

$$\tilde{\psi}(x) = 2b \sum_{j=-n}^{n} \left(\frac{\cos\left(a^{-2j+c}\left(a+\frac{1}{a}\right)\pi x\right)\sin\left(a^{-2j+c}\left(a-\frac{1}{a}\right)\pi x\right)}{\pi x} \right), x \neq 0.$$

Final remarks

- The new type of dual windows
- The *Daubechies polynomials*
  - New methods to construct dual pairs of Gabor- and wavelet frames.
- *Meyer scaling functions*
  - Construction of smooth windows generating pairs of dual Gabor frames with additional desirable properties.
  - Construction of an explicitly given smooth window generating a dual frame for a class of wavelet frames.

Main reference:

L.H. Christiansen, O. Christensen, *Construction of smooth compactly supported windows generating dual pairs of Gabor frames*.
To appear in *Asian-European Journal of Mathematics*, 2012.