

Basic relations valid for the Bernstein space B_σ^2
and their extensions to functions from larger spaces
in terms of their distances from B_σ^2

Part 3: Distance functional approach of Part 2 applied to
fundamental theorems of Part 1

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New Trends and Directions in Harmonic Analysis,
Fractional Operator Theory, and Image Analysis

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Approximate sampling theorem (AST) — recall Lecture 1

Theorem (Weiss 1963, Brown 1967, Butzer-Splettstößer 1977)

Let $f \in F^2 \cap S_{\pi/\sigma}^2$ with $\sigma > 0$. Then

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma}{\pi}t - k\right) + (R_\sigma^{\text{WKS}}f)(t) \quad (t \in \mathbb{R}).$$

We have

$$|(R_\sigma^{\text{WKS}}f)(t)| \leq \sqrt{\frac{2}{\pi}} \int_{|v| \geq \sigma} |\hat{f}(v)| dv = \sqrt{\frac{2}{\pi}} \operatorname{dist}_1(f, B_\sigma^2) = o(1).$$

Corollary

If $f \in F^2$, then one has for any $r \in \mathbb{N}$, $\sigma > 0$ and $t \in \mathbb{R}$ the
derivative free error estimate

$$|(R_\sigma^{\text{WKS}}f)(t)| \leq \sqrt{\frac{2}{\pi}} \operatorname{dist}_1(f, B_\sigma^2) \leq c \int_\sigma^\infty \omega_r(f, v^{-1}, L^2(\mathbb{R})) \frac{dv}{\sqrt{v}}.$$

If in addition $f \in \operatorname{Lip}_r(\alpha, L^2(\mathbb{R}))$ for $1/2 < \alpha \leq r$, then

$$(R_\sigma^{\text{WKS}}f)(t) = \mathcal{O}(\sigma^{-\alpha+1/2}) \quad (\sigma \rightarrow \infty)$$

uniformly in $t \in \mathbb{R}$.

Corollary

Let $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$ for some $r \in \mathbb{N}$. Then for $t \in \mathbb{R}$,

$$|(R_\sigma^{\text{WKS}} f)(t)| \leq c \sigma^{-r+1/2} \|f^{(r)}\|_{L^2(\mathbb{R})} \quad (\sigma > 0).$$

If moreover $f^{(r)} \in \text{Lip}_1(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \leq 1$, then

$$(R_\sigma^{\text{WKS}} f)(t) = \mathcal{O}(\sigma^{-r-\alpha+1/2}) \quad (\sigma \rightarrow \infty).$$

Or, if $f^{(r)} \in M_*^{2,1}$, then

$$(R_\sigma^{\text{WKS}} f)(t) = \mathcal{O}(\sigma^{-r}) \quad (\sigma \rightarrow \infty).$$

Corollary

If $f \in H^2(\mathcal{S}_d)$, then for $t \in \mathbb{R}$,

$$|(R_\sigma^{\text{WKS}} f)(t)| \leq c_d e^{-d\sigma} \|f\|_{H^2(\mathcal{S}_d)} \quad (\sigma > 0).$$

Function spaces, continued

Hardy spaces:

$$H^p(\mathcal{S}_d) := \left\{ f : f \text{ analytic on } \{z \in \mathbb{C} : |\Im z| < d\}, \right.$$

$$\left. \|f\|_{H^p(\mathcal{S}_d)} := \left\{ \sup_{0 < y < d} \int_{\mathbb{R}} \frac{|f(t-iy)|^p + |f(t+iy)|^p}{2} dt \right\}^{1/p} < \infty \right\}.$$

Wiener amalgam space — modulation space:

$$M^{2,1} := \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : \sum_{n \in \mathbb{Z}} \frac{1}{h} \left\{ \int_n^{n+1} \left| f\left(\frac{v}{h}\right) \right|^2 dv \right\}^{1/2} < \infty, h > 0 \right\},$$

$$M_*^{2,1} := \left\{ f \in M^{2,1} : \text{series converges uniformly in } h \text{ on bounded subintervals of } (0, \infty) \right\}.$$

Function spaces

$$F^2 := \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \widehat{f} \in L^1(\mathbb{R}) \right\},$$

$$S_h^p := \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : (f(hk))_{k \in \mathbb{Z}} \in \ell^p(\mathbb{Z}) \right\}.$$

Lipschitz spaces:

$$\omega_r(f; \delta; L^2(\mathbb{R})) := \sup_{|h| \leq \delta} \left\| \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f(u+jh) \right\|_{L^2(\mathbb{R})},$$

$$\text{Lip}_r(\alpha; L^2(\mathbb{R})) := \{f \in L^2(\mathbb{R}) : \omega_r(f; \delta; L^2(\mathbb{R})) = \mathcal{O}(\delta^\alpha), \delta \rightarrow 0+\}.$$

Sobolev spaces:

$$W^{r,p}(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : f \in \text{AC}_{\text{loc}}^{r-1}(\mathbb{R}), f^{(k)} \in L^p(\mathbb{R}), 0 \leq k \leq r \right\}.$$

Reproducing kernel formula (RKF)

Theorem (extended, Butzer et al. 2011)

Let $f \in F^2 \cap S_{\sigma/\pi}^2$, $\sigma > 0$. Then

$$f(t) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \text{sinc}\left(\frac{\sigma}{\pi}(t-u)\right) du + (R_\sigma^{\text{RKF}} f)(t) \quad (t \in \mathbb{R}),$$

$$(R_\sigma^{\text{RKF}} f)(t) := \frac{1}{\sqrt{2\pi}} \int_{|v| > \sigma} \widehat{f}(v) e^{itv} dv.$$

Furthermore,

$$|(R_\sigma^{\text{RKF}} f)(t)| \leq \frac{1}{\sqrt{2\pi}} \int_{|v| > \sigma} |\widehat{f}(v)| dv = \frac{1}{\sqrt{2\pi}} \text{dist}_1(f, B_\sigma^2) = o(1).$$

Three corollaries above are also valid here.

General Parseval decomposition formula (GDPF)

Theorem (Butzer–Gessinger 1995/97)

Let $f \in F^2 \cap S_{\pi/\sigma}^1$ and $g \in F^2$. Then for $\sigma > 0$

$$\int_{\mathbb{R}} f(u) \overline{g(u)} du = \frac{\pi}{\sigma} \sum_{k \in \mathbb{Z}} f\left(\frac{\pi k}{\sigma}\right) \overline{g\left(\frac{\pi k}{\sigma}\right)} + R_{\sigma}(f, g),$$

$$R_{\sigma}(f, g) = \int_{\mathbb{R}} (R_{\sigma}^{\text{WKS}} f)(u) \overline{g(u)} du - \sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \int_{|v| \geq \sigma} \widehat{g}(v) e^{ik\pi v/\sigma} dv.$$

$$|R_{\sigma}(f, g)| \leq \|R_{\sigma}^{\text{WKS}} f\|_{L^2} \|g\|_{L^2} + \sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \sum_{k \in \mathbb{Z}} \left| f\left(\frac{k\pi}{\sigma}\right) \right| \int_{|v| > \sigma} |\widehat{g}(v)| dv.$$

Theorem

Let $f \in F^2 \cap S_{\pi/\sigma}^1$ and $g \in F^2$. Then for $\sigma > 0$

$$\int_{\mathbb{R}} f(u) \overline{g(u)} du = \frac{\pi}{\sigma} \sum_{k \in \mathbb{Z}} f\left(\frac{\pi k}{\sigma}\right) \overline{g\left(\frac{\pi k}{\sigma}\right)} + R_{\sigma}(f, g),$$

$$|R_{\sigma}(f, g)| \leq \|R_{\sigma}^{\text{WKS}} f\|_{L^2} \|g\|_{L^2} + \sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \sum_{k \in \mathbb{Z}} \left| f\left(\frac{k\pi}{\sigma}\right) \right| \int_{|v| > \sigma} |\widehat{g}(v)| dv.$$

If $f, g \in W^{1,2}(\mathbb{R}) \cap C(\mathbb{R})$, then

$$|R_{\sigma}(f, g)| \leq \frac{C}{\sigma} \left\{ \text{dist}_2(f', B_{\sigma}^2) + \text{dist}_2(g', B_{\sigma}^2) + \frac{\pi}{\sigma} \text{dist}_2(f', B_{\sigma}^2) \text{dist}_2(g', B_{\sigma}^2) \right\}.$$

Recall Lecture 2

If $f \in W^{1,2}(\mathbb{R}) \cap C(\mathbb{R})$ with $v\widehat{f}(v) \in L^1(\mathbb{R})$, then for each $r \in \mathbb{N}$,

$$\text{dist}_2(f', B_{\sigma}^2) \leq c \left\{ \int_{\sigma}^{\infty} [\omega_r(f', v^{-1}, L^2(\mathbb{R}))]^2 dv \right\}^{1/2}.$$

If in addition $f' \in \text{Lip}_r(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \leq r$, then

$$\text{dist}_2(f', B_{\sigma}^2) = \mathcal{O}(\sigma^{-1-\alpha}) \quad (\sigma \rightarrow \infty).$$

Corollary

If $f, g \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$, $r \geq 2$, and $f^{(r)}, g^{(r)} \in \text{Lip}_1(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \leq 1$, then

$$R_{\sigma}(f, g) = \mathcal{O}(\sigma^{-r-\alpha}) \quad (\sigma \rightarrow \infty).$$

If instead $f^{(r)}, g^{(r)} \in M_{*}^{2,1}$, then

$$R_{\sigma}(f, g) = \mathcal{O}(\sigma^{-r-1/2}) \quad (\sigma \rightarrow \infty).$$

Corollary

If $f, g \in H^2(\mathcal{S}_d)$, then

$$R_{\sigma}(f, g) \leq c \exp(-\sigma d) \|f\|_{H^2(\mathcal{S}_d)} \|g\|_{H^2(\mathcal{S}_d)} \quad (\sigma > 0).$$

Approximate sampling theorem for derivatives

Theorem

For $f \in F^2 \cap S_{\pi/\sigma}^2$, $\sigma > 0$, with $v^s \widehat{f}(v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for some $s \in \mathbb{N}_0$. Then

$$f^{(s)}(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}^{(s)}\left(\frac{\sigma}{\pi}\left(t - \frac{k\pi}{\sigma}\right)\right) + (R_{s,\sigma}^{\text{WKS}} f)(t) \quad (t \in \mathbb{R}),$$

$$(R_{s,\sigma}^{\text{WKS}} f)(t) := \frac{1}{\sqrt{2\pi}} \int_{|v| > \sigma} \widehat{f}(v) \left[(iv)^r e^{ivt} - [(iv)^r e^{ivt}]^* \right] dv.$$

$[(iv)^s e^{ivt}]^*$ is the 2π -periodic extens. of $(iv)^s e^{ivt}$ from $(-\pi, \pi)$ to \mathbb{R} . Furthermore,

$$|(R_{s,\sigma}^{\text{WKS}} f)(t)| \leq \sqrt{\frac{2}{\pi}} \int_{|v| \geq \sigma} |v^s \widehat{f}(v)| dv = \sqrt{\frac{2}{\pi}} \operatorname{dist}_1(f^{(s)}, B_\sigma^2) = o(1)$$

uniformly for $t \in \mathbb{R}$ for $\sigma \rightarrow \infty$.

Corollary

Let $s \in \mathbb{N}$, $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$ for some $r \geq s + 1$. Then for $t \in \mathbb{R}$,

$$|(R_{s,\sigma}^{\text{WKS}} f)(t)| \leq c \sigma^{-r+s+1/2} \|f^{(r)}\|_{L^2(\mathbb{R})} \quad (\sigma > 0).$$

If moreover $f^{(r)} \in \operatorname{Lip}_1(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \leq 1$, then

$$(R_{s,\sigma}^{\text{WKS}} f)(t) = \mathcal{O}(\sigma^{-r-\alpha+s+1/2}) \quad (\sigma \rightarrow \infty).$$

Corollary

If $f \in H^2(\mathcal{S}_d)$, then for $t \in \mathbb{R}$,

$$|(R_{s,\sigma}^{\text{WKS}} f)(t)| \leq c_d \sigma^s e^{-d\sigma} \|f\|_{H^2(\mathcal{S}_d)} \quad (\sigma > 0).$$

Corollary

If $f \in F^2$, $s \in \mathbb{N}$, and $v^s \widehat{f}(v) \in L^1(\mathbb{R})$, then for any $r \in \mathbb{N}$,

$$|(R_{s,\sigma}^{\text{WKS}} f)(t)| \leq \sqrt{\frac{2}{\pi}} \operatorname{dist}_1(f^{(s)}, B_\sigma^2) \leq c \int_\sigma^\infty \omega_r(f^{(s)}, v^{-1}, L^2(\mathbb{R})) \frac{dv}{\sqrt{v}}.$$

If in addition $f^{(s)} \in \operatorname{Lip}_r(\alpha, L^2(\mathbb{R}))$ for $1/2 < \alpha \leq r$, then

$$(R_{s,\sigma}^{\text{WKS}} f)(t) = \mathcal{O}(\sigma^{-\alpha+1/2}) \quad (\sigma \rightarrow \infty).$$

Boas formula for the first derivative

In 1937 Boas established a differentiation formula which may be presented as follows:

Let $f \in B_\sigma^\infty$, where $\sigma > 0$. Then, for $h = \pi/\sigma$, we have

$$f'(t) = \frac{1}{h} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k+1}}{\pi(k - \frac{1}{2})^2} f\left(t + h\left(k - \frac{1}{2}\right)\right).$$

Boas-type formulae for higher derivatives (Schmeisser 2009)

The Boas-type formulae to be established will be deduced as applications of the Whittaker-Kotel'nikov-Shannon sampling theorem for higher order derivatives.

Theorem (Boas-type formulae)

Let $f \in B_\sigma^\infty$ for some $\sigma > 0$, and $s \in \mathbb{N}$. Then

$$f^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k=-\infty}^{\infty} (-1)^{k+1} A_{s,k} f\left(t + h\left(k - \frac{1}{2}\right)\right) \quad (t \in \mathbb{R}),$$

$$A_{s,k} := \frac{(2s-1)!}{\pi\left(k - \frac{1}{2}\right)^{2s}} \sum_{j=0}^{s-1} \frac{(-1)^j}{(2j)!} [\pi\left(k - \frac{1}{2}\right)]^{2j} \quad (k \in \mathbb{Z}),$$

the series being absolutely and uniformly convergent.

A similar expansion holds for even order derivatives.

Theorem (Boas-type formulae, extended to non-bandlimited funct.)

Let $s \in \mathbb{N}$, $f \in F^2$ and let $v^{2s-1}\widehat{f}(v) \in L^1(\mathbb{R})$. Then $f^{(2s-1)}$ exists and for $h > 0$, $\sigma := \pi/h$ there holds

$$f^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{s,k} f\left(t + h\left(k - \frac{1}{2}\right)\right) + (R_{2s-1,\sigma} f)(t),$$

$$\begin{aligned} |(R_{2s-1,\sigma} f)(t)| &\leq \sqrt{\frac{2}{\pi}} \int_{|v| \geq \sigma} |v^{2s-1}\widehat{f}(v)| \, dv \\ &= \sqrt{\frac{2}{\pi}} \operatorname{dist}_1(f^{(2s-1)}, B_\sigma^2). \end{aligned}$$

The corollaries stated above for the remainder of the approximate sampling theorem are also valid for $R_{2s-1,\sigma} f$.

Proof

Assume $\sigma = \pi$. Setting $t = 1/2$ in derivative sampling theorem yields

$$f^{(2s-1)}\left(\frac{1}{2}\right) = \sum_{k=-\infty}^{\infty} f(k) \operatorname{sinc}^{(2s-1)}\left(\frac{1}{2} - k\right).$$

The sinc-terms can be evaluated by Leibnitz' rule, namely,

$$\operatorname{sinc}^{(2s-1)}\left(\frac{1}{2} - k\right) = \underbrace{\frac{(-1)^{k+1}(2s-1)!}{\pi\left(k - \frac{1}{2}\right)^{2s}} \left\{ \sum_{j=0}^{s-1} \frac{(-1)^j [\pi\left(k - \frac{1}{2}\right)]^{2j}}{(2j)!} \right\}}_{=(-1)^{k+1} A_{s,k}}.$$

$$f^{(2s-1)}\left(\frac{1}{2}\right) = \sum_{k=-\infty}^{\infty} f(k) (-1)^{k+1} A_{s,k}.$$

For arbitrary $\sigma > 0$, $t \in \mathbb{R}$ apply this to $u \mapsto f(hu + t - h)$. \square

Theorem (Bernstein inequality)

For $f \in B_\sigma^p$, $1 \leq p \leq \infty$, $\sigma > 0$, there holds

$$\|f^{(s)}\|_{L^p(\mathbb{R})} \leq \sigma^s \|f\|_{L^p(\mathbb{R})} \quad (s \in \mathbb{N}).$$

Corollary (Bernstein inequality for trigonometric polynomials)

If $f(t) = t_n(t) = \sum_{k=-n}^n c_k e^{ikt}$, a trigonometric polynomial of degree n , then $t_n \in B_n^\infty$, and

$$\|t_n^{(s)}\|_{L^\infty(\mathbb{R})} \leq n^s \|t_n\|_{L^\infty(\mathbb{R})} \quad (s \in \mathbb{N}).$$

Theorem (Extended Bernstein inequality)

Let $s \in \mathbb{N}$, $f \in F^2$ and suppose that $v^s \widehat{f}(v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as a function of v . Then, for any $\sigma > 0$, we have

$$\|f^{(s)}\|_{L^2(\mathbb{R})} \leq \sigma^s \|f\|_{L^2(\mathbb{R})} + \operatorname{dist}_2(f^{(s)}, B_\sigma^2).$$

The proofs follow from the Boas-type formulae above.

The Hilbert transform

Hilbert transform or conjugate function of $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, is defined by Cauchy principal value

$$\tilde{f}(t) := \lim_{\delta \rightarrow 0^+} \int_{|u| > \delta} \frac{f(t-u)}{u} du = \text{PV} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t-u)}{u} du,$$

It defines a bounded linear operator from $L^2(\mathbb{R})$ into itself, and $[\tilde{f}]^\wedge(v) = (-i \operatorname{sgn} v) \hat{f}(v)$ a. e. If $v^s \hat{f}(v) \in L^1(\mathbb{R})$, then by Fourier inversion formula,

$$[\tilde{f}]^{(s)}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(v) (-i \operatorname{sgn} v) (iv)^s e^{ivt} dv \quad (t \in \mathbb{R}).$$

Thus $[\tilde{f}]^{(s)} = \widetilde{f^{(s)}}$; i. e. derivation and taking Hilbert transform are commutative operations.

Since $\operatorname{sinc}^\sim(v) = 1/\sqrt{2\pi}$ for $|v| \leq \pi$ and $= 0$ otherwise,

$$\operatorname{sinc}^\sim(t) = \frac{1 - \cos \pi t}{\pi t} = \frac{\sin^2\left(\frac{\pi t}{2}\right)}{\frac{\pi t}{2}}.$$

Theorem (Boas formulae for the Hilbert transform, extended vers.)

Let $s \in \mathbb{N}$, $f \in F^2$ and let $v^{2s-1} \hat{f}(v) \in L^1(\mathbb{R})$. Then $f^{(2s-1)}$ exists and for $h > 0$, $\sigma := \pi/h$,

$$\tilde{f}^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k \in \mathbb{Z}} (-1)^{k+1} \tilde{A}_{s,k} f(t+hk) + (\tilde{R}_{2s-1,\sigma} f)(t),$$

where

$$\begin{aligned} |(\tilde{R}_{2s-1,\sigma} f)(t)| &\leq \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \sigma} |v^{2s-1} \hat{f}(v)| dv \\ &= \frac{1}{\sqrt{2\pi}} \operatorname{dist}_1(f^{(2s-1)}, B_\sigma^2). \end{aligned}$$

Theorem (Boas-type formulae for the Hilbert transform)

Let $f \in B_\sigma^2$, $\sigma > 0$ and $h := \pi/\sigma$. Then for $s \in \mathbb{N}$,

$$\tilde{f}^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k=-\infty}^{\infty} (-1)^{k+1} \tilde{A}_{s,k} f(t+hk) \quad (t \in \mathbb{R}).$$

$$\tilde{A}_{s,0} := (-1)^s \frac{\pi^{2s-1}}{2s},$$

$$\tilde{A}_{s,k} := \frac{(2s-1)!}{\pi k^{2s}} \left\{ (-1)^k - \sum_{j=0}^{s-1} \frac{(-1)^j}{(2j)!} (\pi k)^{2j} \right\} \quad (k \neq 0).$$

The proof is based on the sampling theorem for the Hilbert transform,

$$\tilde{f}^{(2s-1)}(0) = \sum_{k=-\infty}^{\infty} f(k) \operatorname{sinc}^\sim(2s-1)(-k).$$

Proof

First assume $h = 1$, i. e., $\sigma = \pi$. Let

$$f_1(t) := \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \pi} \hat{f}(v) e^{itv} dv.$$

Then $f - f_1 \in B_\pi^\infty$ and so the bandlimited case applies to this difference, i. e.,

$$(\tilde{R}_{2s-1,\pi} f)(t) = (\tilde{R}_{2s-1,\pi}(f-f_1))(t) + (\tilde{R}_{2s-1,\pi} f_1)(t) = (\tilde{R}_{2s-1,\pi} f_1)(t),$$

and we find that for $t = 0$,

$$(\tilde{R}_{2s-1,\pi} f)(0) = \tilde{f}_1^{(2s-1)}(0) - \sum_{k=-\infty}^{\infty} (-1)^{k+1} \tilde{A}_{s,k} f_1(k).$$

Proof, continued

$$(\tilde{R}_{2s-1,\pi}f)(0) = \tilde{f}_1^{(2s-1)}(0) - \sum_{k=-\infty}^{\infty} (-1)^{k+1} \tilde{A}_{s,k} \tilde{f}_1(k).$$

$$\tilde{f}_1^{(2s-1)}(0) = \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \pi} \hat{f}(v) (-i \operatorname{sgn} v) (iv)^{2s-1} dv$$

$$f_1(k) = \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \pi} \hat{f}(v) e^{ikv} dv \quad (k \in \mathbb{Z}).$$

Inserting the last two equations into the first one and interchanging summation and integration yields

$$(\tilde{R}_{2s-1,\pi}f)(0) = \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \pi} \hat{f}(v) \left[(-i \operatorname{sgn} v) (iv)^{2s-1} - \sum_{k=-\infty}^{\infty} (-1)^{k+1} \tilde{A}_{s,k} e^{ikv} \right] dv.$$

Applications

Consider $g(t) := 1/(1+t^2)$, $t \in \mathbb{R}$, Fourier transform $\sqrt{\pi/2} \exp(-|v|)$, Hilbert transform $\tilde{g}(t) = t/(1+t^2)$. Extended sampling theorem for Hilbert transform takes on concrete form for \tilde{g}' ,

$$\left| \frac{1-t^2}{(1+t^2)^2} \right|$$

$$- \sum_{k=-\infty}^{\infty} \frac{\sigma^2}{\sigma^2 + (k\pi)^2} \frac{\pi(\sigma t - k) \sin(\pi(\sigma t - k)) + \cos(\pi(\sigma t - k)) - 1}{\pi(\sigma t - k)^2}$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \sigma} \sqrt{\frac{\pi}{2}} |v| e^{-|v|} dv = (1+\sigma)e^{-\sigma} \quad (\sigma > 0).$$

Proof, continued

The foregoing infinite series is a Fourier series of a 2π -periodic function and can be evaluated to be

$$\sum_{k=-\infty}^{\infty} (-1)^{k+1} \tilde{A}_{s,k} e^{ikv} = (-1)^{s+1} |v|^{2s-1} \quad (|v| < \pi).$$

$$|(\tilde{R}_{2s-1,\pi}f)(0)|$$

$$= \left| \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \pi} \hat{f}(v) \left[(-i \operatorname{sgn} v) (iv)^{2s-1} - (-1)^{s+1} [|v|^{2s-1}]^* \right] dv \right|$$

$$\leq \sqrt{\frac{2}{\pi}} \int_{|v| \geq \sigma} |v|^{2s-1} |\hat{f}(v)| dv = \sqrt{\frac{2}{\pi}} \operatorname{dist}_1(f^{(2s-1)}, B_\sigma^2).$$

$[|v|^{2s-1}]^*$ is the 2π -periodic extension of $|v|^{2s-1}$ from $(-\pi, \pi)$ to \mathbb{R} .

This is the desired inequality for $\sigma = \pi$, $h = 1$ and $t = 0$. For the general case apply this estimate to the function $u \mapsto f(t + hu)$. \square

Truncation error

In practise one has to deal with finite sum, no infinite series. Leads to additional *truncation error*,

$$(T_{\sigma,N}f)(t)$$

$$= \sum_{|k| \geq N+1} \frac{\sigma^2}{\sigma^2 + (k\pi)^2} \frac{\pi(\sigma t - k) \sin(\pi(\sigma t - k)) + \cos(\pi(\sigma t - k)) - 1}{\pi(\sigma t - k)^2}$$

This yields for the truncation error

$$\begin{aligned} |(T_{\sigma,N}f)(t)| &\leq \frac{\sigma^2(2\gamma+1)}{\pi^2(\gamma-1)} \sum_{|k| \geq N+1} \frac{1}{|k|^3} \\ &\leq \frac{2\sigma^2(2\gamma+1)}{\pi^2(\gamma-1)} \int_N^\infty \frac{1}{u^3} du = \frac{\sigma^2(2\gamma+1)}{\pi^2(\gamma-1)} N^{-2}. \end{aligned}$$

for $N \geq \gamma\sigma|t|$ and some constant $\gamma > 1$.

Combined error

Combining the aliasing error with truncation error we finally obtain

$$\left| \frac{1-t^2}{(1+t^2)^2} - \sum_{k=-N}^N \text{clear} \right| \leq (1+\sigma)e^{-\sigma} + \frac{\sigma^2(2\gamma+1)}{\pi^2(\gamma-1)} N^{-2}$$

$$(\sigma > 0; N \geq \gamma\sigma|t|).$$

Similarly, Boas-type theorem for derivative \tilde{g}' takes the form

$$\left| \frac{1-t^2}{(1+t^2)^2} - \left\{ \frac{\pi}{2h} \frac{1}{1+t^2} - \frac{1}{h} \sum_{k=-\infty}^{\infty} \frac{2}{\pi(2k+1)^2} \frac{1}{1+[t+(2k+1)h]^2} \right\} \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \sigma} \sqrt{\frac{\pi}{2}} |v| e^{-|v|} dv = (1+\sigma)e^{-\sigma} \quad (\sigma > 0).$$

For the second order derivative \tilde{g}'' one obtains

$$\left| \frac{2t^3-6t}{(1+t^2)^3} - \frac{1}{h^2} \sum_{k=-\infty}^{\infty} \frac{8(\pi-2\pi k+2(-1)^k)}{\pi(2k-1)^3} \frac{(-1)^k}{1+[t+(2k+1)h]^2} \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \sigma} \sqrt{\frac{\pi}{2}} v^2 e^{-|v|} dv = (2+2\sigma+\sigma^2)e^{-\sigma} \quad (\sigma > 0).$$

These are the aliasing errors for the reconstruction of derivatives of the Hilbert transform in terms of Boas-type formulae. In both cases the truncation errors can be handled in a similar fashion as above.

Some Boas-type formulae for bandlimited functions and their Hilbert transform

$$f'(t) = \frac{1}{h} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k+1}}{\pi(k-\frac{1}{2})^2} f\left(t+h\left(k-\frac{1}{2}\right)\right),$$

$$f''(t) = -\frac{\pi^2}{3h^2} f(t) + \frac{2}{h^2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} f(t+hk) \frac{(-1)^{k+1}}{k^2},$$

$$f^{(3)}(t) = \frac{1}{h^3} \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+16}}{\pi(\frac{1}{2}-k)^4} \left[1 - \frac{\pi^2}{2} \left(\frac{1}{2}-k\right)^2 \right] f\left(t+h\left(k-\frac{1}{2}\right)\right),$$

$$\tilde{f}'(t) = \frac{\pi}{2h} f(t) + \frac{1}{h} \sum_{k=-\infty}^{\infty} \frac{-2}{\pi(2k+1)^2} f(t+h(2k+1)),$$

$$\tilde{f}''(t) = \frac{1}{h^2} \sum_{k=-\infty}^{\infty} (-1)^{k+1} \frac{2[(-1)^k + (\pi(k-\frac{1}{2}))]}{\pi(k-\frac{1}{2})^3} f\left(t+h\left(k-\frac{1}{2}\right)\right).$$

References I




 P. L. Butzer, P. J. S. G. Ferreira, J. R. Higgins, G. Schmeisser, and R. L. Stens.

The sampling theorem, Poisson's summation formula, general Parseval formula, reproducing kernel formula and the Paley-Wiener theorem for bandlimited signals - their interconnections.
Applicable Analysis, 90(3-4):431-461, 2011.




 P. L. Butzer, P. J. S. G. Ferreira, J. R. Higgins, G. Schmeisser, and R. L. Stens.

The generalized Parseval decomposition formula, the approximate sampling theorem, the approximate reproducing kernel formula, Poisson's summation formula and Riemann's zeta function; their interconnections for non-bandlimited functions.
to appear.



References II

-  P. L. Butzer, P. J. S. G. Ferreira, G. Schmeisser, and R. L. Stens.
The summation formulae of Euler-Maclaurin, Abel-Plana, Poisson, and their interconnections with the approximate sampling formula of signal analysis.
Results Math., 59(3-4):359–400, 2011.
-  P. L. Butzer and A. Gessinger.
The approximate sampling theorem, Poisson's sum formula, a decomposition theorem for Parseval's equation and their interconnections.
Ann. Numer. Math., 4(1-4):143–160, 1997.
-  P. L. Butzer, G. Schmeisser, and R. L. Stens.
Basic relations valid for the Bernstein space B_σ^p and their extensions to functions from larger spaces in terms of their distances from B_σ^p .
to appear.

References III

-  P. L. Butzer, G. Schmeisser, and R. L. Stens.
Shannon's sampling theorem for bandlimited signals and their Hilbert transform, Boas-type formulae for higher order derivatives — the aliasing error involved by their extensions from bandlimited to non-bandlimited signals.
invited for publication in Entropy.
-  P. L. Butzer and W. Splettstösser.
A sampling theorem for duration limited functions with error estimates.
Inf. Control, 34:55–65, 1977.
-  P. L. Butzer, W. Splettstösser, and R. L. Stens.
The sampling theorem and linear prediction in signal analysis.
Jber.d.Dt.Math.-Verein., 90:1–70, 1988.

References IV

-  G. Schmeisser.
Numerical differentiation inspired by a formula of R. P. Boas.
J. Approximation Theory, 160:202–222, 2009.
-  R. L. Stens.
A unified approach to sampling theorems for derivatives and Hilbert transforms.
Signal Process., 5(2):139–151, 1983.