Study of an example of multifractal and "sparse" signal

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Introduction

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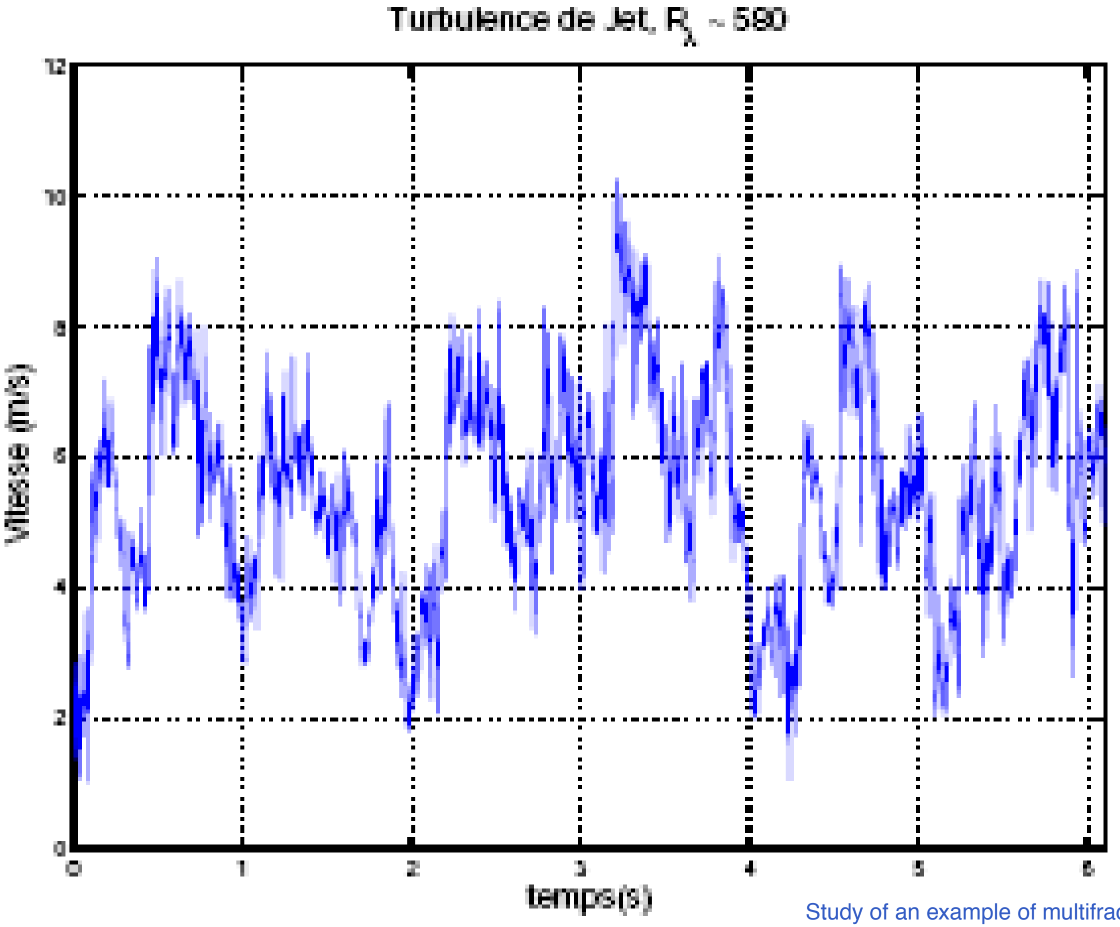

Required tools

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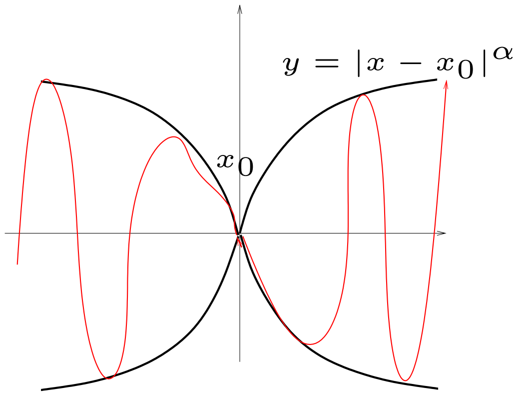
- a numerical method to compute the "size" of the sets of points with a given pointwise regularity

$\rightarrow$ Frisch-Parisi formula (1985), Wavelet Transform Maxima Method (Arnéodo and all, 1989), Wavelet-leaders method (Jaffard and all 2002...).

Pointwise Hölder exponent:



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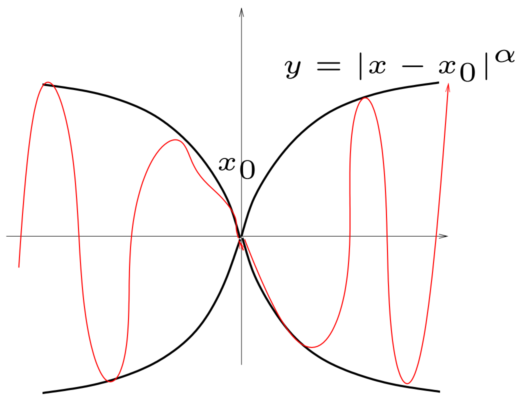

Definition: Let $x_0 \in \mathbb{R}^d$ and $\alpha \geq 0$.
A locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $C^\alpha(x_0)$ if there exists $C > 0$ and a polynomial P_{x_0} with $deg(P) \leq [\alpha]$ and such that on a neighborhood of x_0,

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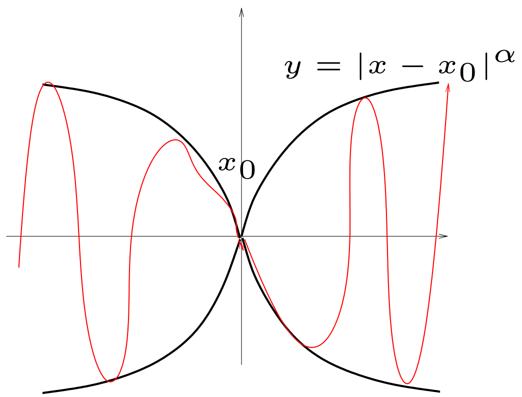

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$\to$ The pointwise Hölder exponent of f at x_0 is
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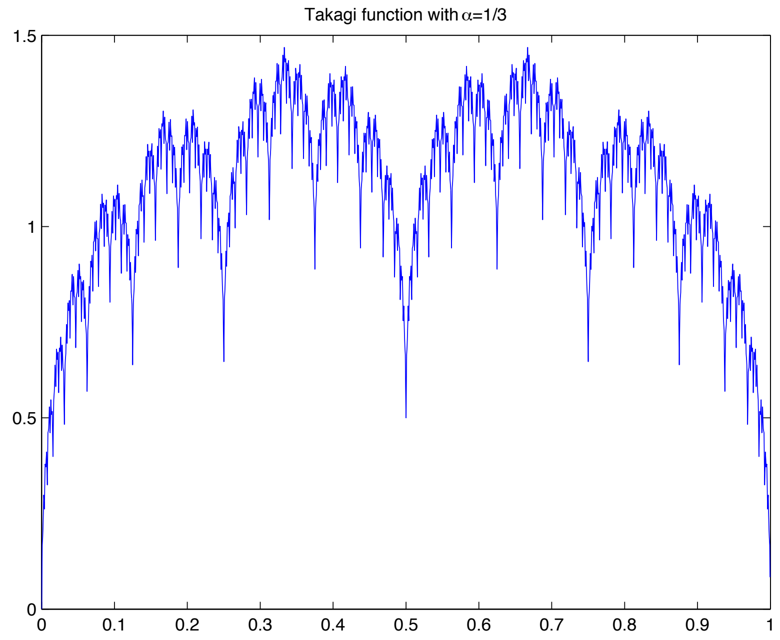

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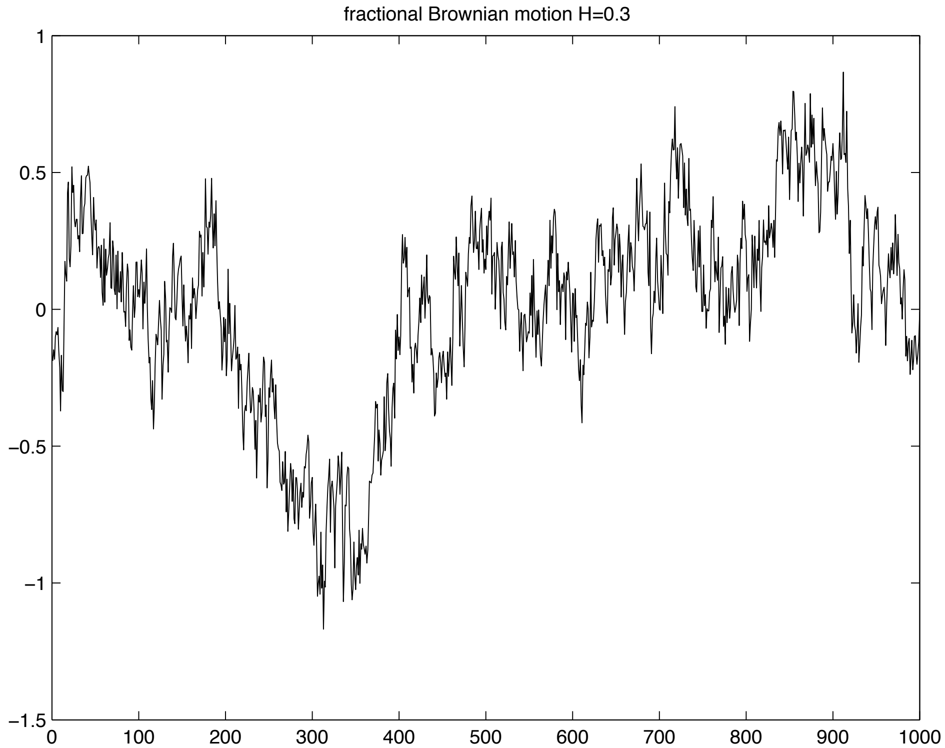

with probability 1 each sample path satisfies $h_f(x_0) = H$ at each x_0

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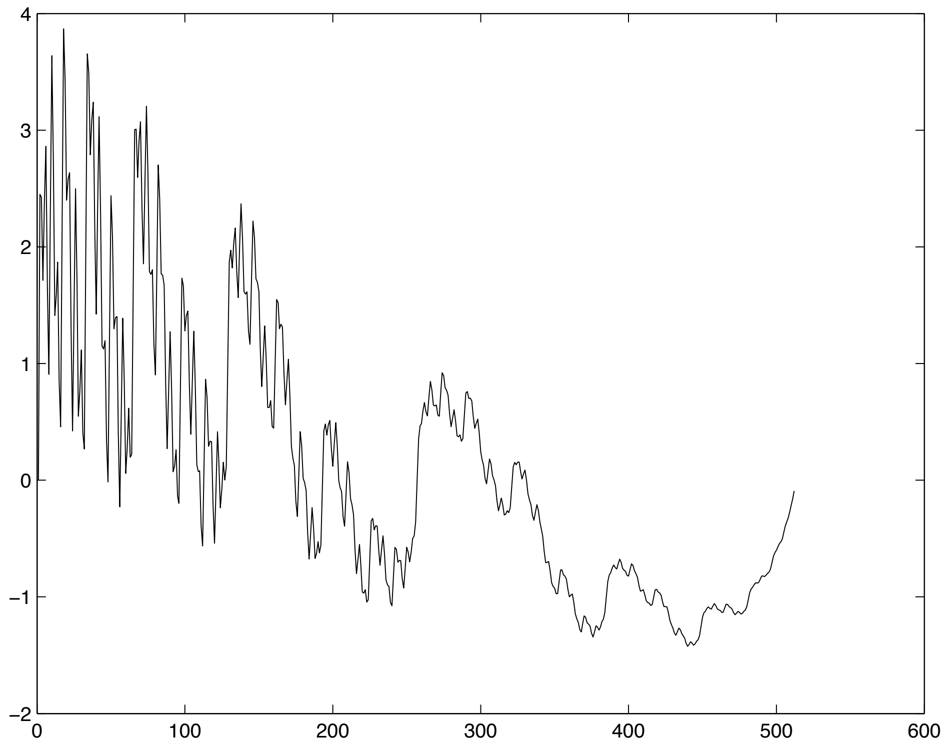

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One first evaluates the “structure function” with f in L^q:

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When y gets to 0, we have $S_q(y) \sim |y|^{\zeta_f(q)}$,
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$$d_f(h) = \inf_q (qh - \zeta_f(q) + d).$$

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→ One can find counterexamples.

Exploring pointwise regularity

The "p-exponent"

Definition:(Calderon and Zygmund 1961)

Let $p \in [1, \infty]$ and u such that $u \geq -\frac{d}{p}$. Let f be a function in L^p_{loc}. f belongs to $T^p_u(x_0)$

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$$\forall \rho \leq R : \left(\frac{1}{\rho^d} \int_{|x-x_0| \leq \rho} |f(x) - P(x)|^p dx \right)^{\frac{1}{p}} \leq C \rho^u. \qquad (2)$$

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$\rightarrow$ the p-exponent of f at x_0 is $u^p_f(x_0) = \sup\{u : f \in T^p_u(x_0)\}$

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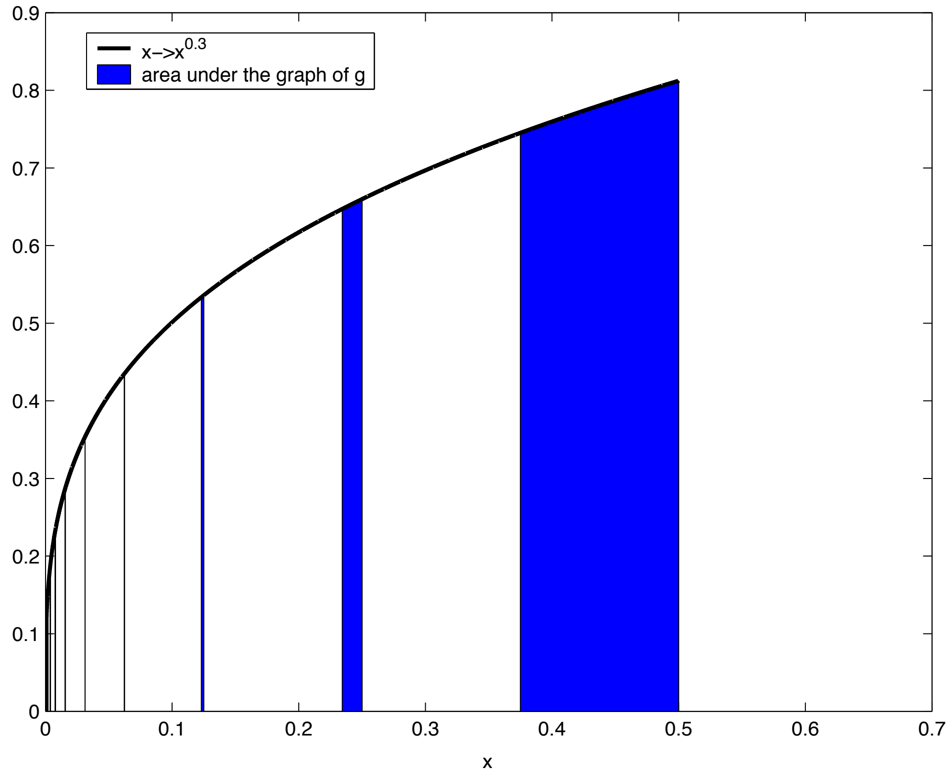
- if $f \in C^h(x_0)$ then $u_f^p(x_0) \geq h$.
- Less straightforward: Bessel potential of order α $\mathcal{J}^\alpha$ (fractional integration operator) maps continuously $T_u^p(x_0)$ to $T_{u+\alpha}^p(x_0)$.

Example:

$D_j = [\,1/2^j - 1/2^{3j},\, 1/2^j\,]$, where $j \geq 0$

$g(x) = |\,x\,|^\alpha \sum_{j=1}^{\infty} I_{D_j}(x).$

$h_g(0) = \alpha < u_g^p(0) = \alpha + 1/p$ for any $p \geq 1$.



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- $\rightarrow$ from the point of view of the Hölder exponent
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- Model: developped by Jaffard in the early 90's with the help of wavelet basis.

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  - we write $\lambda = (j, k) = [\frac{k}{2^j}, \frac{k+1}{2^j}]$ which yields $\psi_\lambda = \psi(2^j . - k)$

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- $\alpha = 1$ in the following and β integer.

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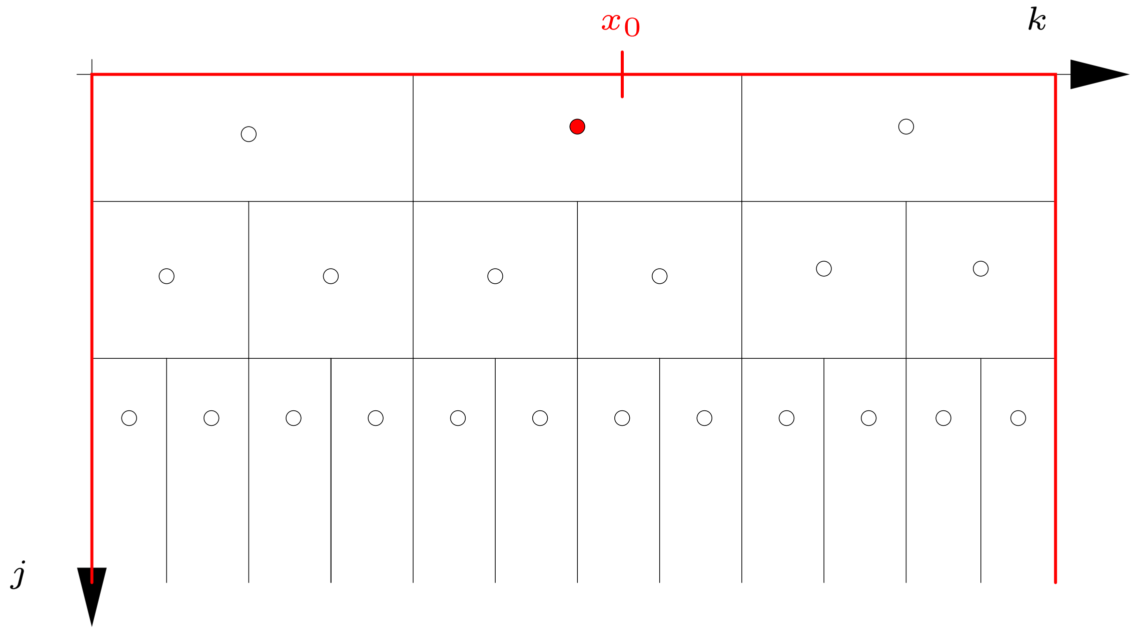

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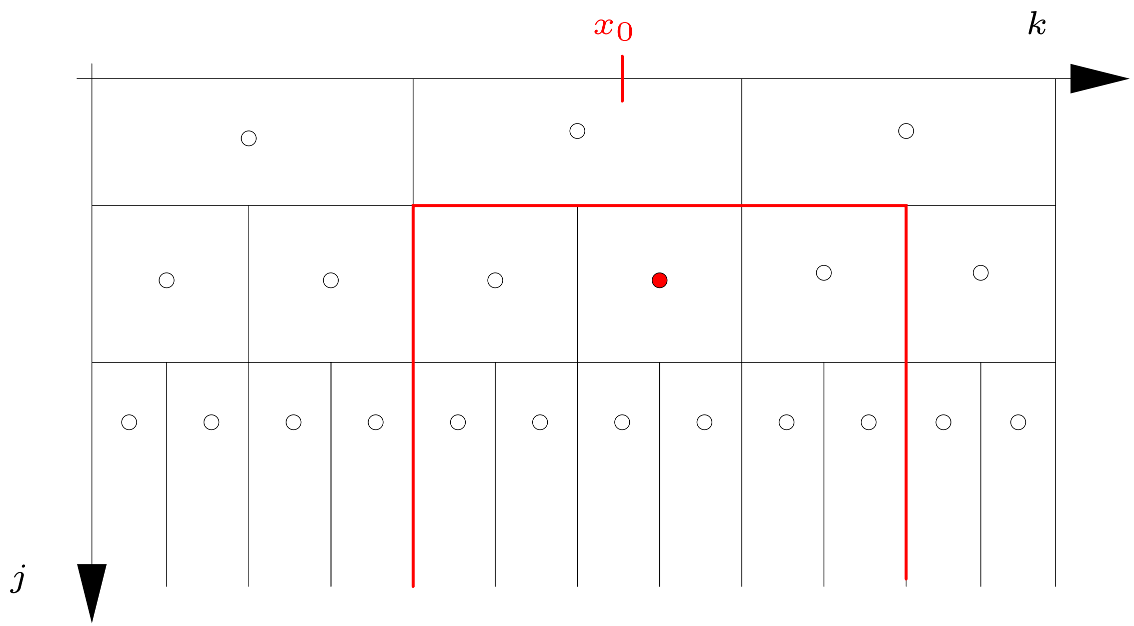
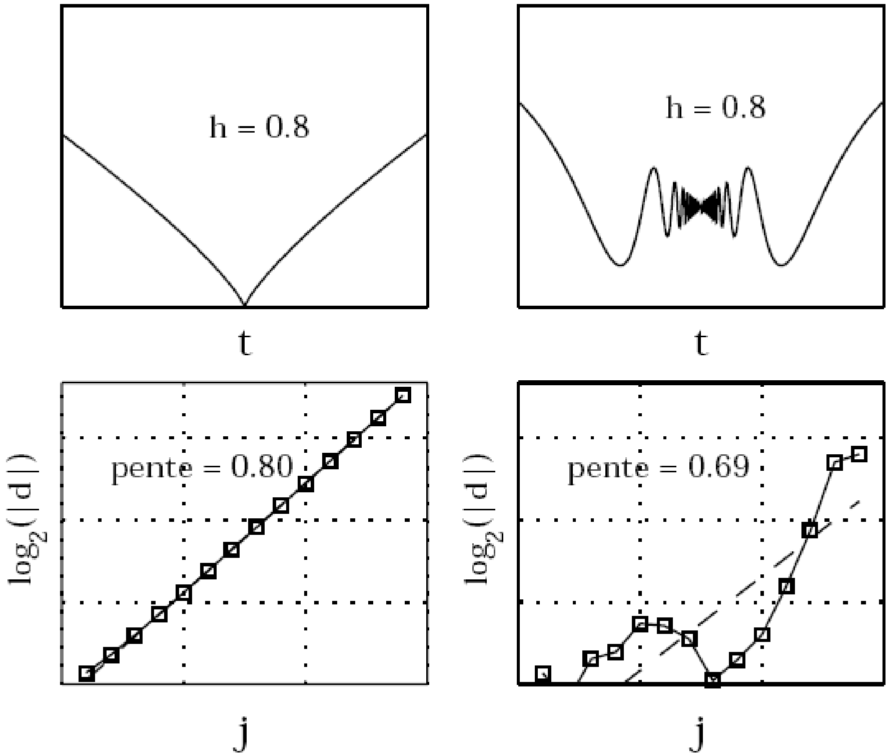


Numerical results



(Abry-Lashermes, 2005)

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- (Jaffard 2004) Let $S_f(j,x_0)(x) = \sqrt{\sum_{\lambda' \subset 3\lambda_j(x_0)} C_\lambda^2 \chi_\lambda(x)}$ then
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Approximation rate by dyadics

Let $x_0 \in \mathbb{R}$ and $f \in L^\infty_{loc}(\mathbb{R})$.

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- The dimension of $E_r = \{x_0 : \; r(x_0) = r\}$ is exactly $\frac{1}{r}$

Spectrum of singularities

Theorem 1 *Let* α*,* β *and* γ*, with* $\alpha = 1$ *and* $\beta \geq 1$ *an integer and* $\gamma > 0$ *a non integer. Let* $p \geq 1$*.*

- *Suppose* $r(x_0) < \beta$ *then* $h_f(x_0) = \frac{\beta\gamma}{r(x_0)}$ *and* $u_p^f(x_0) = \frac{\beta(\gamma+\frac{1}{p})}{r(x_0)}$

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- *Suppose* $r(x_0) \geq \beta$ *then* $h_f(x_0) = \beta\gamma$ *and* $u_p^f(x_0) = \beta(\gamma + \frac{1}{p})$*.*

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- *Suppose $r(x_0) < \beta$ then $h_f(x_0) = \frac{\beta\gamma}{r(x_0)}$ and $u_p^f(x_0) = \frac{\beta(\gamma+\frac{1}{p})}{r(x_0)}$*
- *Suppose $r(x_0) \geq \beta$ then $h_f(x_0) = \beta\gamma$ and $u_p^f(x_0) = \beta(\gamma + \frac{1}{p})$.*
- *f satisfies a multifractal type formula and its spectrum of singularities is $d_f(h) = \frac{h \, \mathbf{1}_{[\gamma,\gamma\beta]}(h)}{\beta\gamma}$.*

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$\rightarrow$ *related to oscillation spaces*

$f \in \mathcal{O}_q^s \Leftrightarrow 2^{sq-1} \sum_{\lambda \in \Lambda_j} d_\lambda^q < \infty$ *with* $d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|$

Multifractal formalism

- Let $S_f(q,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^q$ with $d_\lambda = \sup\limits_{\lambda' \subset \lambda} |c_{\lambda'}|$

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  let compute $S_f(q,j)$ as $j \to +\infty$ with the help of the scaling function f defined by

$$\eta_f(q) = \liminf_{j\to+\infty} \left(\frac{\log\left(S_f(q,j)\right)}{\log(2^{-j})} \right); \tag{3}$$

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- If $f \in C^\delta(\mathbb{R}^n)$ the multifractal formalism claims $d_f(u) = \inf_q (uq - \eta_f(q) + d)$

Multifractal formalism

Let the p-leader: $d_{\lambda,p} = \left(\sum_{\lambda' \subset \lambda} 2^{-d(j'-j)} |c_{\lambda'}|^p\right)^{1/p}$.

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we estimate the decreasing of $S_f(p,q,j)$ when $j \to +\infty$ with the help of the p-*scaling function* of f defined by

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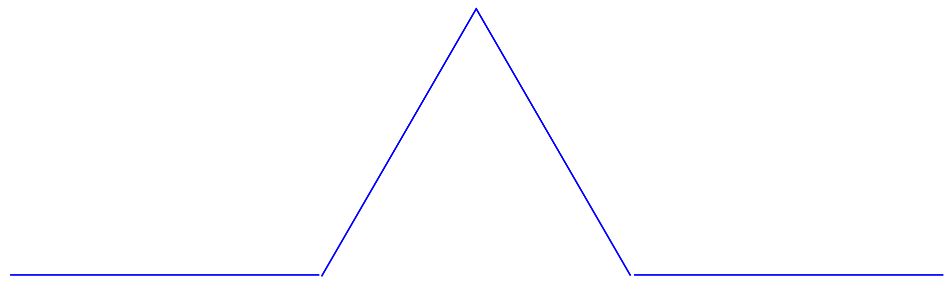
the **multifractal formalism for the p-exponent claims:**

$$d_f^p(u) = \inf_q (uq - \eta_f(p,q) + d),$$

Notion of dimension:

Example: Koch’s snowflake

Construction:



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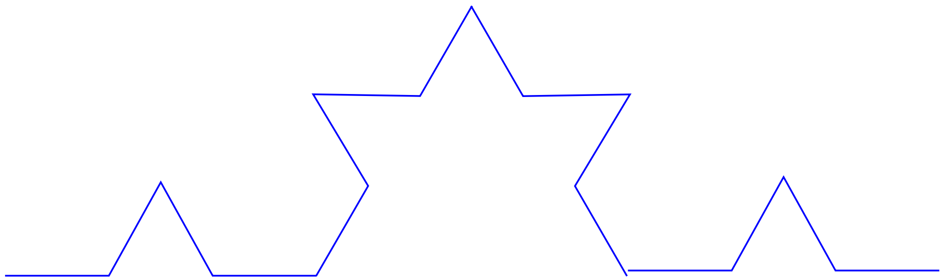

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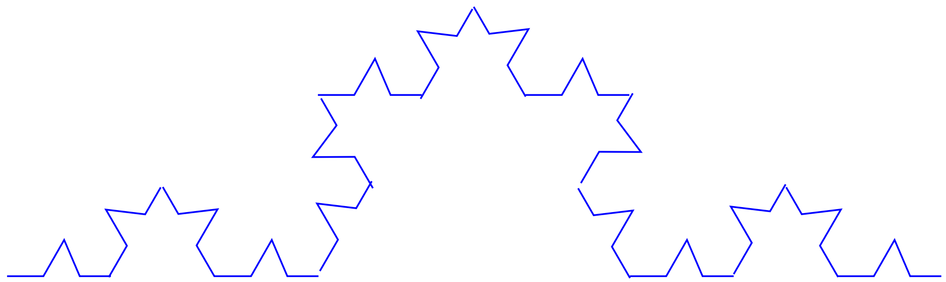

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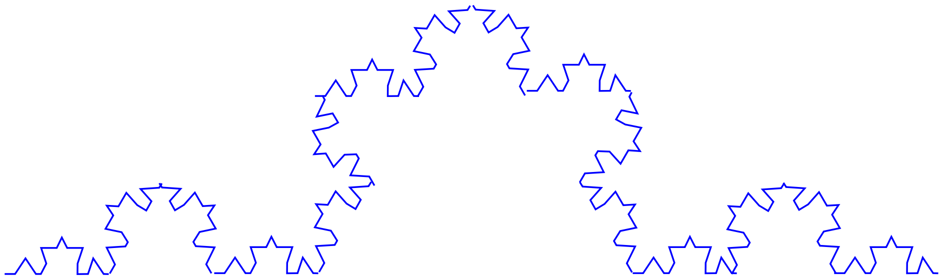
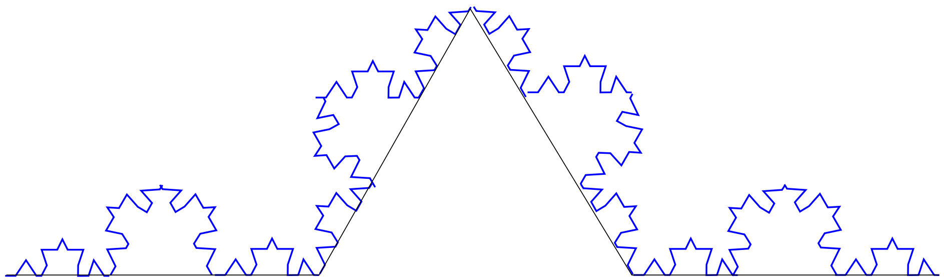


Notion of dimension:

Covering by 4 segments of length $1/3$



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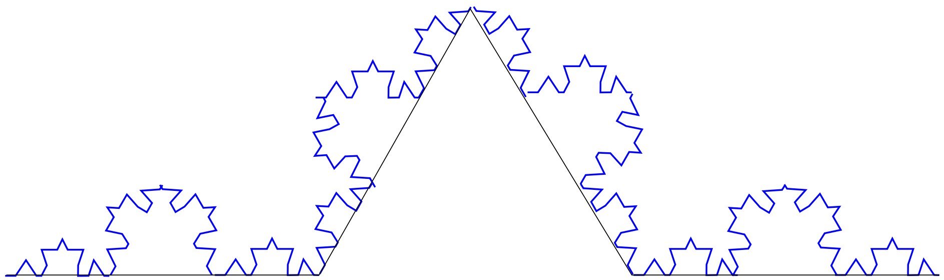
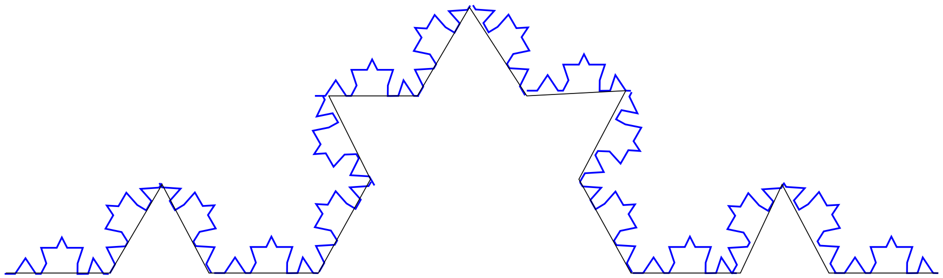


The length of the covering is $\frac{4}{3}$

Notion of dimension:

Covering by 4^2 segments of length $1/3^2$



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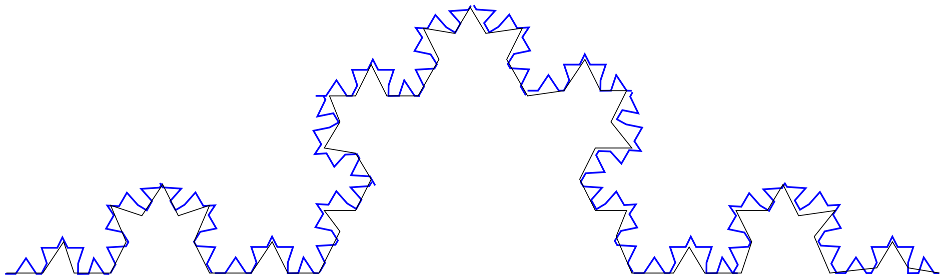
Notion of dimension:

Covering by 4^3 segments of length $1/3^3$



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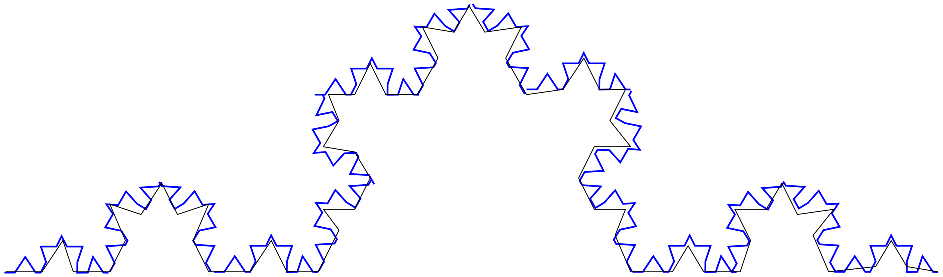


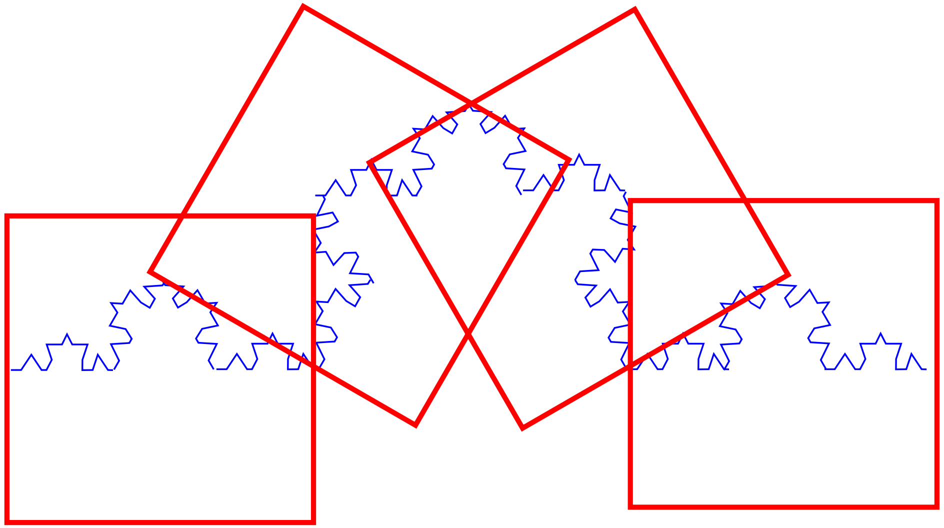
The length of the covering is $\frac{4^3}{3^3}$.

At step n the length of the covering is $\frac{4^n}{3^n}$.

And so the total length is not finite.

Notion of dimension:

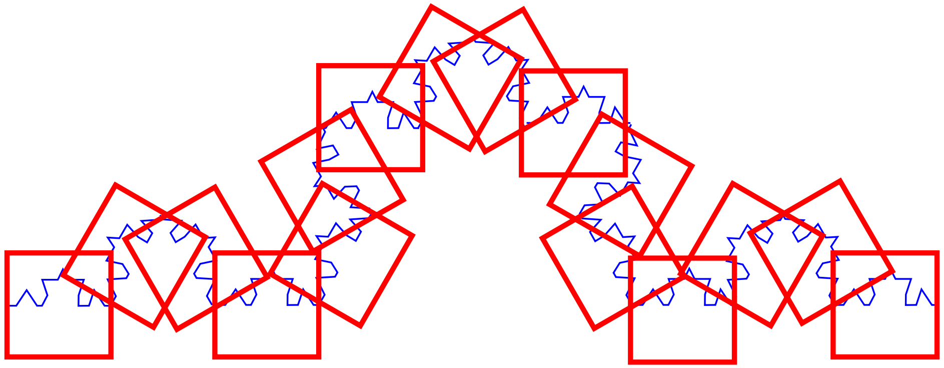
Covering by 4 square of size $1/3$



The area of the covering is $\frac{4}{3^2}$

Notion of dimension :

Covering by 16 square of size $1/3^2$



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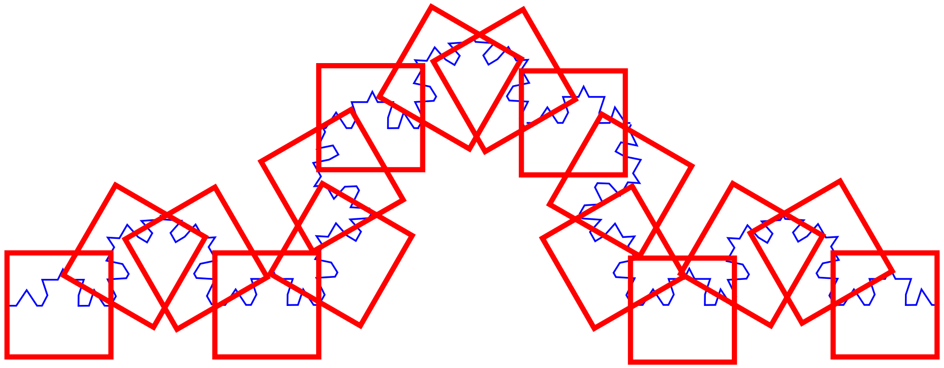

The area of the covering is $\frac{16}{3^4}$

At step n the area of the covering is $\frac{4^n}{3^{2n}}$.

And so the total area is zero.

Hausdorff dimension:

Let d such that $0 < \lim_{n \to +\infty} \frac{4^n}{3^{dn}} < +\infty$.

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Hausdorff measure

Definition 1 *Let $F \subset \mathbb{R}^d$ and $s \geq 0$.*

$\forall \delta > 0$, we denote

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : F \subset \bigcup_i U_i, diam(U_i) \leq \delta \right\}$$

where $diam(U_i)$ means the diameter of U_i.

The s-dimensional Hausdorff measure of F is $\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F)$.

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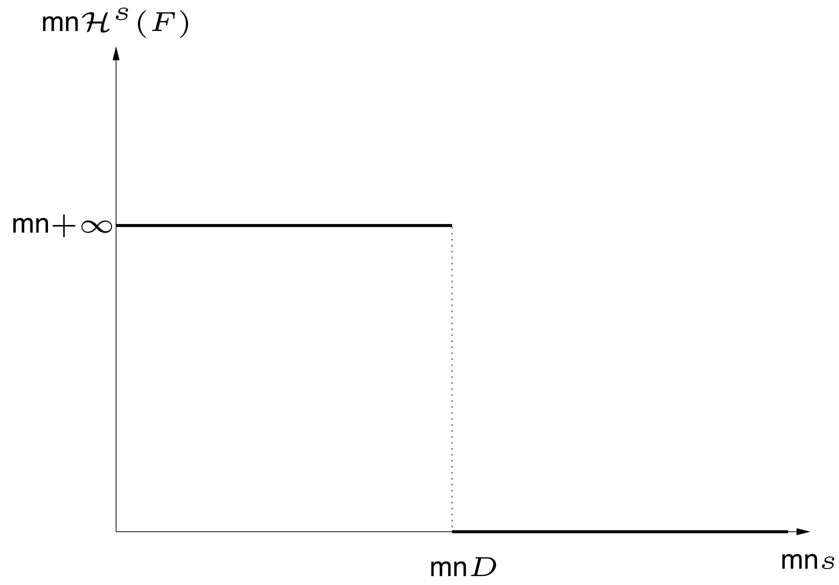

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- Very difficult to compute numerically for one set.
- Impossible to compute when you have an infinity of sets !