Adapted curvelet sparse regularization in limited angle tomography

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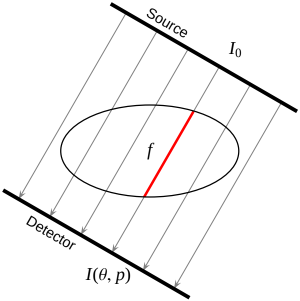
Radon Transform:

$$\mathcal{R}f(\theta, p) = \int_{\mathbb{R}} f(p\theta + t\theta^{\perp})\, \mathrm{d}t$$

$$= \ln\left(\frac{I_0}{I(\theta, p)}\right).$$



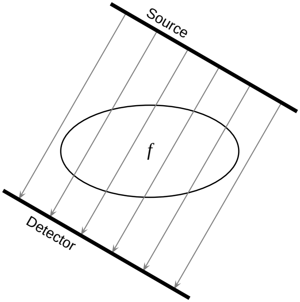
Reconstruction problem:

- Given noisy measurements

$$y^{\delta} = \mathcal{R}f + \eta, \quad \delta > 0,\ \|\eta\|_2^2 < \delta.$$

- Find a good approximation to f.

Notation: $p \in \mathbb{R}$, $\theta = (\theta_1, \theta_2) \in S^1$ and $\theta^{\perp} = (-\theta_2, \theta_1)$.



"**available** projections"

- **Limited angular range**, i.e.,

$$\theta_1, \ldots, \theta_L \in S^1_\Phi = \left\{\theta \in S^1 : \ (\cos\theta, \sin\theta)^T, \ |\varphi| \le \Phi\right\}, \quad \Phi < \frac{\pi}{2}.$$

- **Limited Angle Radon Transform:**

$$\boxed{\mathcal{R}_\Phi f := \mathcal{R} f|_{Z_\Phi}, \quad Z_\Phi = S^1_\Phi \times \mathbb{R}.}$$

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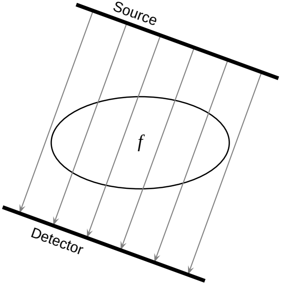

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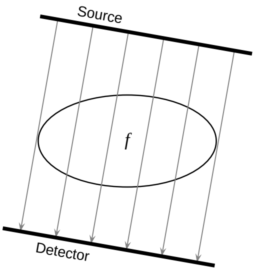

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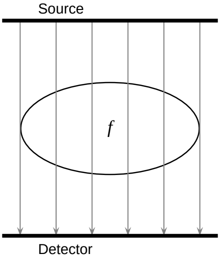

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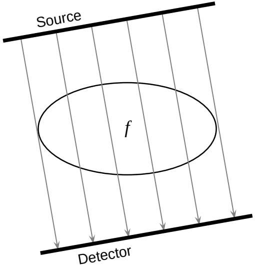

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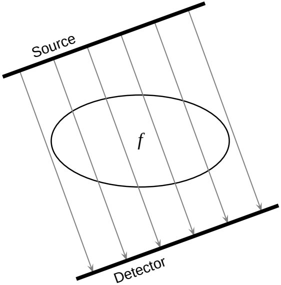

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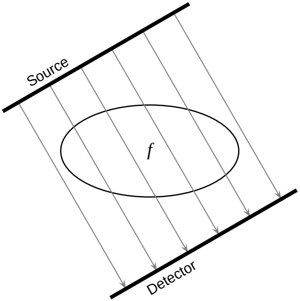

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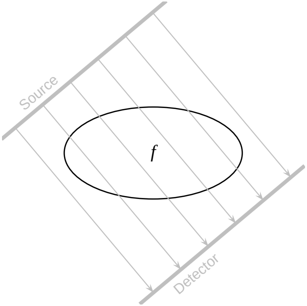

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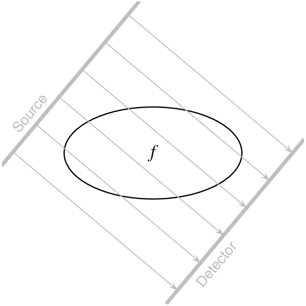

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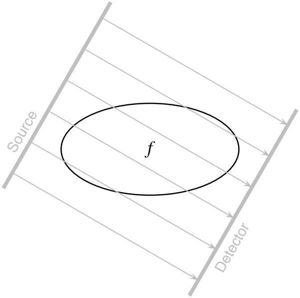

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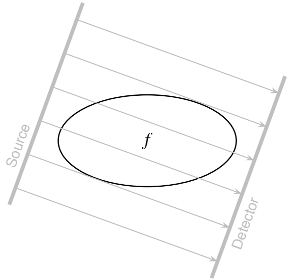

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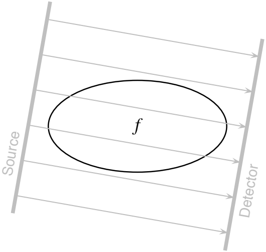

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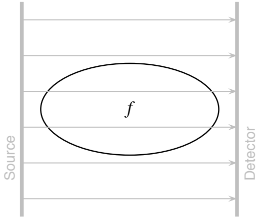

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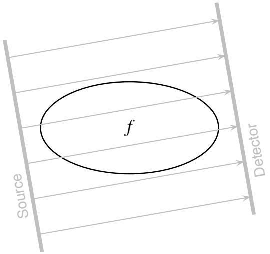

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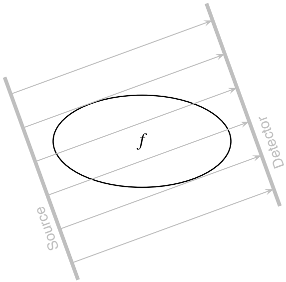

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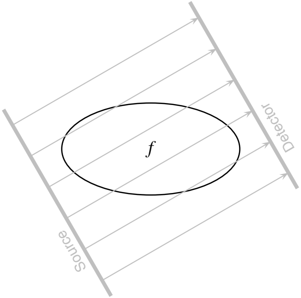

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Problems

- Data highly incomplete.
- Reconstruction problem $y^\delta = \mathcal{R}_\Phi f + \eta$ is serverely ill-posed, i.e., solution operator $\mathcal{R}_\Phi^{-1}$ is not continuous.
- Classical reconstruction algorithms, such as filtered backprojection (FBP), do not perform well.
- Only specific features (visible singularities) of the original object can be reconstructed reliably.

Topics and Tools

- Reconstruction & Fast algorithms
- Ill-posedness & Regularization
- Microlocal analysis & Singularity detection
- ...

Authors

Natterer, Louis, Quinto, Ramm, Madych, Nelson, Davison, Herman, ...

Design a reconstruction method which is

1. stable,
2. edge-preserving,
3. is adapted to the limited angle geometry.

Recall: Solution operator $\mathcal{R}_\Phi^{-1}$ is not continuous

Definition

A **regularization method** (for $\mathcal{R}_\Phi^{-1}$) is a family of continuous operators $\{\mathcal{S}_\alpha\}_{\alpha>0}\,,\ \mathcal{S}_\alpha : \operatorname{ran}(\mathcal{R}_\Phi) \to \operatorname{dom}(\mathcal{R}_\Phi)$ such that

$$\alpha = \alpha(\delta) \to 0 \text{ as } \delta \to 0$$

and

$$\lim_{\delta\to 0} \mathcal{S}_{\alpha(\delta)} y^\delta = \mathcal{R}_\Phi^{-1} y \quad \forall y \in \operatorname{ran}(\mathcal{R}_\Phi)\,,$$

for all $y \in \operatorname{ran}(\mathcal{R}_\Phi)$ and all y^δ with $\left\|y - y^\delta\right\|_2^2 < \delta$.

α = regularization parameter (or parameter choice rule)

$\mathcal{S}_\alpha$ = regularization operator

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How can one construct regularization methods?

α = regularization parameter (or parameter choice rule)

$\mathcal{S}_\alpha$ = regularization operator

Strategy

Integration of additional **a-priori information** about the solution into the reconstruction process.

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Classical approach

Enforce smoothness of the solution:

$$f^{\alpha,\delta} := \mathcal{S}_\alpha y^\delta := \arg\min_f \left\{ \left\| \mathcal{R}_\Phi f - y^\delta \right\|_2^2 + \alpha \left\| f \right\|_{\text{smoothness}} \right\}.$$

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Example: SOBOLEV NORM

$$\|f\|_{\text{smoothness}} = \|f\|_\alpha := \left(\int \left| \hat{f}(\xi) \right|^2 (1 + |x|^2)^\alpha \, \mathrm{d}\xi \right)^{\frac{1}{2}}$$

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Example: BESOV NORM

$$\|f\|_{\text{smoothness}} = \|f\|_{p,q,\alpha} := \|f\|_p + \left(\int_0^\infty [t^{-\alpha} \omega_p^r(f,t)]^q \, \mathrm{d}t \right)^{\frac{1}{q}},$$

where $r = \lfloor \alpha \rfloor + 1$ and $\omega_p^r(f,t)$ is the r-th modulus of smoothness of f.

?, ...

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Example: TOTAL VARIATION NORM

$$\|f\|_{\text{smoothness}} = \|f\|_{TV} := \int |\nabla f(x)| \, \mathrm{d}x$$

Prototype for edge-preserving reconstruction.

?, **?**, ...

Strategy

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Sparse regularization

Enforce sparsity of the solution with respect to a dictionary $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$:

$$\hat{c} = \arg\min_c \left\{ \left\| \mathcal{R}_\Phi T^* c - y^\delta \right\|_2^2 + \|c\|_{\ell^1_w} \right\},$$

$$\hat{f} = T^*\hat{c} = \sum_{i\in\mathcal{I}} \hat{c}_i \psi_i.$$

sparse: only a few entries of the coefficient vector $\hat{c}$ are non-zero.

(Daubechies et al., 2004)

c = coefficient vector w.r.t. dictionary $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$,

T^* = sythesis operator w.r.t. dictionary Ψ, i.e., $T^* : c \mapsto \sum_{i\in\mathcal{I}} c_i\psi_i$.

How to choose an appropriate dictionary Ψ?

- A-priori information about the **solution should be sparsely encoded by** Ψ.
- Optimally **sparse representation of edges** w.r.t. Ψ.
- **Directionality** to adapt to the limited angle geometry.

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Idea: Curvelets

- Highly directional.
- Optimally sparse for "C^2 functions with C^2 edges", **?**.

- Generating curvelets are defined in the Fourier domain

$$\widehat{\psi}_{-1,0,0}(r,\omega) = W_0(r), \qquad \widehat{\psi}_{j,0,0}(r,\omega) = 2^{-3j/4} \cdot W(2^{-j}r) \cdot V\left(\frac{2^{\lceil j/2\rceil+1}}{\pi}\omega\right).$$

- Curvelets $\left\{\psi_{j,l,k}\right\}_{(j,l,k)\in\mathcal{I}}$ by translation and rotation

$$\psi_{-1,0,k}(x) = \psi_{-1,0,0}(x-k), \qquad \psi_{j,l,k}(x) = \psi_{j,0,0}(\varrho_{\theta_{j,l}}(x-b_k^{j,l})).$$

- Curvelet index set $\mathcal{I} = \left\{(j,l,k) : \ j \in \mathbb{N}_0, -2^{\lceil j/2\rceil+1} \le l < 2^{\lceil j/2\rceil+1}, k \in \mathbb{Z}^2\right\}$ has a 3-parameter structure:

j = **scale**, l = **orientation**, k = **location**.

(r,ω) = polar coordinates in the frequency domain,

ϱ_θ = 2D rotation matrix w.r.t. angle θ,

$V(\omega)$ = real valued, smooth function with $\operatorname{supp} V \subset (-1,1)$, (angular window)

$W(r)$ = smooth function with $W \ge 0$, $\operatorname{supp} W \subset (1/2, 2)$, (radial window)

$W_0^2(r) = 1 - \sum_{j=0}^{\infty} W^2(2^{-j}r)$.

Both, V and W are assumed to satisfy some admissibilty conditions (**?**).

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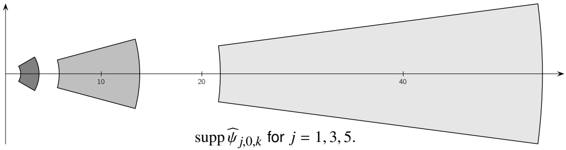
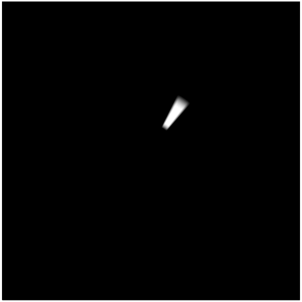
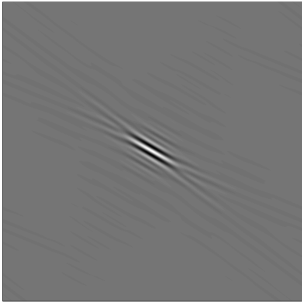


supp $\widehat{\psi}_{j,0,k}$ for $j = 1, 3, 5$.



(a) Frequency domain



(b) Spatial domain

Matlab generated images of the curvelet $\psi_{5,5,0}$.

- **Tight frame** for for $L^2(\mathbb{R}^2)$, i.e., $\forall f \in L^2(\mathbb{R}^2)$:

$$f = \sum_{(j,l,k)\in\mathcal{I}} \left\langle \psi_{j,l,k}, f \right\rangle \psi_{j,l,k}, \qquad \|f\|^2_{L^2(\mathbb{R}^2)} = \sum_{(j,l,k)\in\mathcal{I}} \left|\left\langle \psi_{j,l,k}, f \right\rangle\right|^2.$$

- **Edge preservation**. Best N-term approximation error

$$\inf_{f_N\in\Sigma_N} \|f - f_N\|_2^2 \le C \cdot N^{-2} \cdot (\log N)^3$$

is optimal for piecewise C^2 functions having discontinuities along C^2 curves.

$\Sigma_N := \left\{\sum_{(j,l,k)\in\mathcal{I}_N} c_{j,l,k}\psi_{j,l,k} :\ \mathcal{I}_N \subset \mathcal{I},\ |\mathcal{I}_N| \le N\right\}$

References: **???**

$$\begin{aligned}\hat{c} &= \arg\min_{c}\left\{\frac{1}{2}\left\|\mathbf{K}c - y^{\delta}\right\|_2^2 + \|c\|_{\ell_w^1}\right\},\\ \hat{c} &= \text{Reconstructed curvelet coefficients,}\\ \hat{f} &= T^*\hat{c} = \sum_{n\in\mathcal{I}} \hat{c}_n\psi_n.\end{aligned}$$

- Measurements $y_m = \mathcal{R}_\Phi T^* c(\theta_m, p_m) = \sum_{n\in\mathcal{I}} c_n \mathcal{R}_\Phi \psi_n$.
- System matrix $\mathbf{K} = (K_{m,n})$ is given by

$$K_{m,n} = \mathcal{R}_\Phi T^* \psi_n(\theta_m, p_m),$$

  for all $1 \le m \le M$ and all $n \in \mathcal{I}$.
- **Problem dimension = $|\mathcal{I}|$.**

  Does not depend on the available angular range.

Theorem (F. 2011)

Let $0 < \Phi < \pi/2$. For $j \in \mathbb{N}_0$ we define the polar wedge $W_{\Phi,j}$ at scale 2^{-j} by

$$W_{\Phi,j} = \left\{ \xi \in \mathbb{R}^2 : \xi = r(\cos\omega, \sin\omega)^T,\ r \in \mathbb{R}, |\omega| < \Phi + \pi 2^{-\lceil j/2 \rceil - 1} \right\}.$$

and the invisible index set of curvelet coefficients $\mathcal{I}$ by

$$\mathcal{I}_\Phi^{\text{invisible}} = \left\{ (j,l,k) \in \mathcal{I} : (\cos\theta_{j,l}, \sin\theta_{j,l})^T \notin W_{\Phi,j} \right\}.$$

Then,

$$\boxed{\mathcal{R}_\Phi \psi_{j,l,k} \equiv 0 \text{ for all } (j,l,k) \in \mathcal{I}_\Phi^{\text{invisible}}.}$$

Curvelets which correspond to missing directions are not visible for the limited angle Radon transform $\mathcal{R}_\Phi$

Theorem (F. 2011)

Let $0 < \Phi < \pi/2$, $y^\delta \in \operatorname{ran}(\mathcal{R}_\Phi)$ *and* $w_k \geq w_0 > 0$. *Then, for*

$$\hat{c} = \arg\min_c \left\{ \left\| \mathcal{R}_\Phi T^* c - y^\delta \right\|_2^2 + \|c\|_{\ell_w^1} \right\},$$

it holds that

$$\boxed{\hat{c}_{j,l,k} = 0 \text{ for all } (j,k,l) \in \mathcal{I}_\Phi^{\text{invisible}},}$$

where $\mathcal{I}_\Phi^{\text{invisible}}$ *is the index set of invisible curvelets and* T^* *denotes the sythesis operator*

$$T^* c = \sum_{(j,l,k)\in\mathcal{I}} c_{(j,l,k)} \psi_{(j,l,k)}.$$

$$\hat{c}_\Phi = \arg\min_c \left\{ \frac{1}{2} \left\| \mathbf{K}^\Phi c - y^\delta \right\|_2^2 + \|c\|_{\ell^1_w} \right\},$$

$$\hat{c}_\Phi = \text{Reconstructed curvelet coefficients,}$$

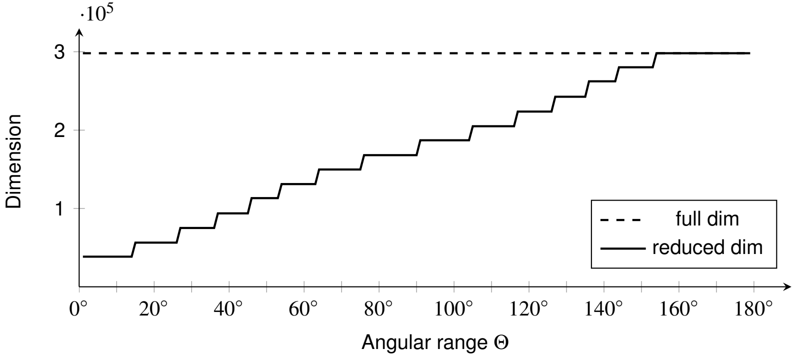
$$\hat{f}_\Phi = T^*\hat{c} = \sum_{n \in \mathcal{I}_\Phi^{\text{visible}}} \hat{c}_n \psi_n.$$

- Adapted system matrix $\mathbf{K}^\Phi = (\mathbf{K}^\Phi_{m,n})$ is given by

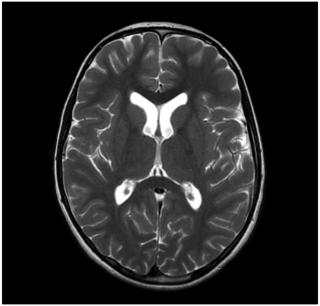
$$\mathbf{K}^\Phi_{m,n} = \mathcal{R}_\Phi T^* \psi_n(\theta_m, p_m), \quad 1 \leq m \leq M, \; n \in \mathcal{I}_\Phi^{\text{visible}}$$

- Adapted problem dimension = $\left|\mathcal{I}_\Phi^{\text{visible}}\right| << |\mathcal{I}|$.

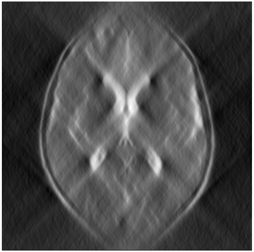
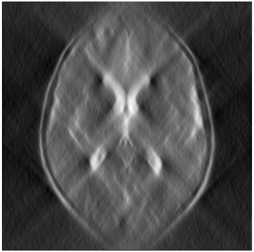
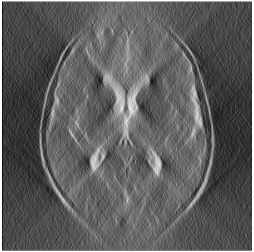
  Highly dependent on the available angular range!



Dimensions of the problem $y^\delta = K_\Phi c + \eta$ in the curvelet domain for an image of size 256×256. The plot shows the dependence of the full dimension – – – and reduced (adapted) dimension ——— on the available **angular range** $[0, \Theta]$.

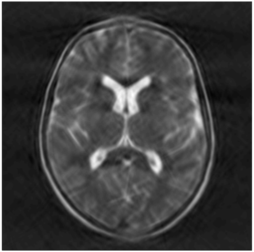
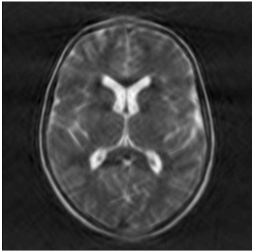


Brainstem testimage

CSR	A – CSR	FBP
		

Reconstruction* (300 iterations) of the Brainstem image of size 300×300:

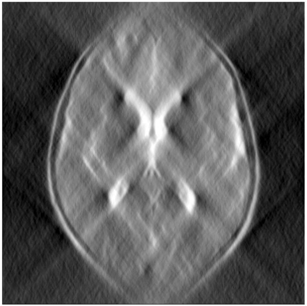
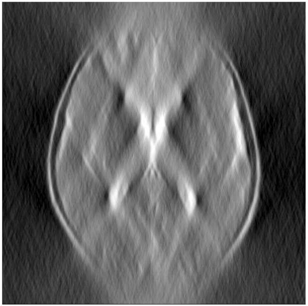
Angular range $[-45^\circ, 45^\circ]$, $\Delta\theta = 1^\circ$, Noiselevel 2%.

CSR	A – CSR	FBP
		

Reconstruction of the Brainstem image of size 300×300:

Angular range $[-80^\circ, 80^\circ]$, $\Delta\theta = 1^\circ$, Noiselevel 2%.

- Curvelet sparse regularization (CSR) for limited angle tomography:
  - Stable reconstruction,
  - Edge-preserving reconstruction.
- Characterization of the null space of the limited angle Radon transform in terms of curvelets & Characterisation of CSR reconstructions.
- Adapted-CSR:
  - dimensionality reduction,
  - better conditioned reduced system matrix,
  - significant speed-up.

A-CSR reconstructions at an angular range $[-45^\circ, 45^\circ]$ and the corresponding artifact reduced version (right).

Thank You!

References

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