Nonnegative Scaling Vectors on the Interval

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Joint work with: David Ruch (Denver) and Yongzhi Yang (UST)

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  - satisfy a dilation equation,
  - reproduce polynomials of the same order reproduced by $\{\phi_{nk}\}$,
  - ensures that the set $S \cup L$ is a Riesz basis for $L^2[0,\infty)$.

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  and assume it generates a Multiresolution Analysis (MRA) for $L^2(\mathbb{R})$.
- ϕ and its integer translates form a partition of unity:

$$\sum_k \phi(t - k) = 1$$

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- We can compute the Fourier transform of $P_r(t)$:

$$\hat{P}_r(\omega) = \frac{1-r^2}{1-2r\cos\omega + r^2}\hat{\phi}(\omega)$$

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Walter and Shen were able to prove that there exists $0 < r_0 < 1$ such that for $r_0 \leq r < 1$, the following hold for P_r:

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$$a_k = \sum_n h_{k-2n} r^{|n|} \frac{1+r^2}{1-r^2} - \frac{r^{|n|+1}}{1-r^2}(h_{k-1-2n} + h_{k+1-2n}).$$

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- $\tilde{P}_r(t) = \frac{1}{2\pi(1-r^2)}((1 + r^2)\phi(t) - r(\phi(t - 1) + \phi(t + 1)))$
- P_r generates the same MRA for $L^2(\mathbb{R})$ as ϕ.

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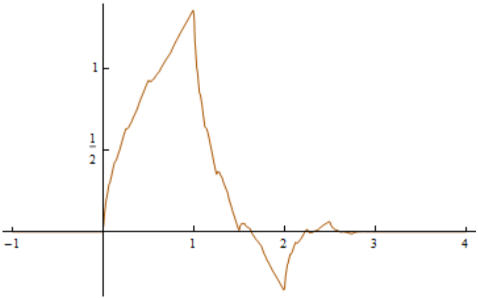
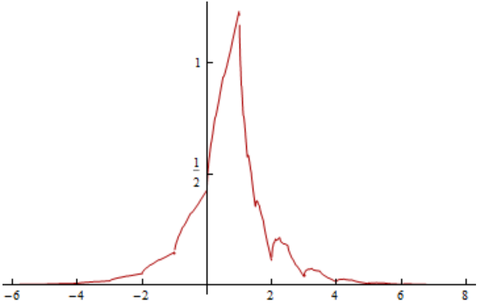
Start with the Daubechies 4-tap orthonormal scaling function $\phi(t)$.



Use $r = -\phi(2)$ to obtain



Note that both orthogonality and compact support are lost.

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- Use a given scaling vector Φ, compactly supported, to create a nonnegative and compactly supported scaling vector from Φ that generates an MRA for $L^2[0, 1]$.
- Preserve polynomial accuracy of the original scaling vector Φ.
- Try not to create too many edge functions.

Definition. Let $\Phi = (\phi^1, \phi^2, \ldots, \phi^A)^T$, $\phi^j \in L^2(\mathbb{R})$ and consider the set

$$V_0 = \overline{\langle \phi^j(\cdot - k) \rangle}_{k \in \mathbb{Z}, \quad j=1,\ldots,A}$$

We say the nested set of spaces

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

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- In this case, Φ satisfies a *matrix refinement equation*

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- We further assume that:
  - Each ϕ^ℓ is compactly supported on $[0, M_\ell]$, $M_\ell \in \mathbb{Z}_+$ and continuous.
  - There is a vector $\mathbf{c} = (c_1, \ldots, c_A)^T$ for which

$$\sum_{\ell=1}^{A} \sum_k c_\ell \phi^\ell(t - k) = \sum_k \mathbf{c} \cdot \Phi(t - k) = 1$$

    This is the partition of unity condition.

Finally, we assume that Φ has polynomial accuracy m. That is, there exist constants f_{nk}^{ℓ} such that for $n = 0, 1, \ldots, m-1$, we have

$$t^n = \sum_{\ell=1}^{A} \sum_{k} f_{nk}^{\ell} \phi^{\ell}(t-k) = \sum_{k} \mathbf{f}_{nk} \cdot \Phi(t-k)$$

Note that from the previous slide

$$1 = t^0 = \sum_{k} \mathbf{c} \cdot \Phi(t-k) = \sum_{k} \mathbf{f}_{0k} \cdot \Phi(t-k)$$

so that $\mathbf{c} = \mathbf{f}_{0k}$ for all $k \in \mathbb{Z}$.

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- are compacted supported,
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- are (anti)symmetric,
- generate an orthonormal basis for V_0.

Example - DGHM. (Donovan, Geronimo, Hardin, Massopust) Take $A = 2$, with the 4-term matrix refinement equation

$$\Phi(t) = \sum_{k=0}^{3} C_k \Phi(2t - k)$$

where

$$C_0 = \begin{bmatrix} 3/5 & 4\sqrt{2}/5 \\ -\sqrt{2}/20 & -3/10 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 3/5 & 0 \\ 9\sqrt{2}/20 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 \\ 9\sqrt{2}/20 & -3/10 \end{bmatrix}, \qquad C_3 = \begin{bmatrix} 0 & 0 \\ -\sqrt{2}/20 & 0 \end{bmatrix}$$

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- $\text{supp}(\phi^1) = [0,1]$ and $\text{supp}(\phi^2) = [0,2]$.
- Achieve polynomial accuracy $m = 2$.
- Form a partition of unity with $c_1 = (1+\sqrt{2})^{-1}$ and $c_2 = \sqrt{2}c_1$.

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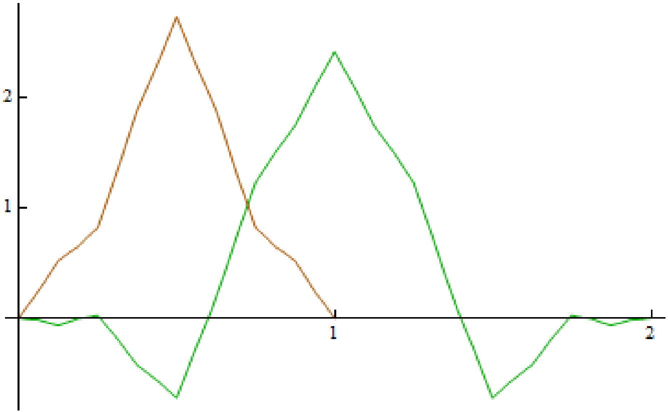
- Form a partition of unity with $c_1 = (1 + \sqrt{2})^{-1}$ and $c_2 = \sqrt{2}c_1$.
- $\phi^1(t) \geq 0$, $t \in \mathbb{R}$.


2
1
1
2

Example. G. Plonka and V. Strela used a two-scale similarity transform in the frequency domain to construct the scaling vector

$$\Phi(t) = \sum_{k=0}^{2} C_k \cdot \Phi(2t - k)$$

where

$$C_0 = \frac{1}{20}\begin{bmatrix} -7 & 15 \\ -4 & 10 \end{bmatrix}, C_1 = \frac{1}{20}\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}, C_2 = \frac{1}{20}\begin{bmatrix} -7 & -15 \\ 4 & 10 \end{bmatrix}$$

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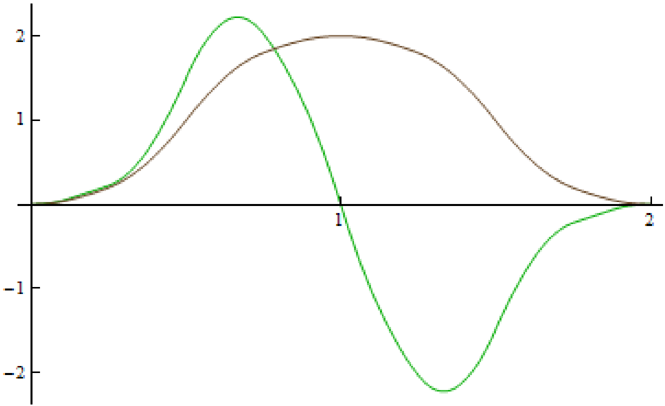
- Φ and its translates are not orthogonal.
- Φ is continuous and has approximation order $m = 3$.
- $\text{supp}\left(\phi^1\right) = \text{supp}\left(\phi^2\right) = [0, 2]$.
- Partition of unity: $c_1 = 0$ and $c_2 = 1$.


2
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- The results of Walter and Shen can be extended to scaling vectors Φ with only a modest condition on the partition of unity coefficients $\mathbf{c}$ of Φ.

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- Moreover, the construction can be altered so that compact support can be retained - that's the contribution of using scaling vectors.

Let $\Phi = (\phi^1, \ldots, \phi^A)$. We say Φ satisfies Condition B if for some $j \in \{1, 2, \ldots, A\}$, $\phi^j(t) \geq 0$ for $t \in \mathbb{R}$ and there exist finite index sets Λ_i and constants c_{ik} for $i \neq j$ such that:

(B1) $\tilde{\phi}^i(t) := \phi^i(t) + \sum_{k \in \Lambda_i} c_{ik}\phi^j(t-k) \geq 0$, $t \in \mathbb{R}$.

(B2) $d_j := c_j - \sum_{i \neq j} \sum_{k \in \Lambda_i} c_i c_{ik} \geq 0$,

(B3) $c_i \geq 0$, for $i \neq j$.

Here $\mathbf{c} = (c_1, \ldots, c_A)^T$ are the coefficients that form the partition of unity for Φ:

$$1 = \sum_k \mathbf{c} \cdot \Phi(t-k)$$

Theorem. (**D. Ruch**, **PVF**) Suppose the scaling vector

$$\Phi = (\phi^1, \phi^2, \dots, \phi^A)^T$$

is bounded, compactly supported, has polynomial accuracy $m \geq 1$, and satisfies Condition B. Then the nonnegative scaling vector

$$\tilde{\Phi} = (\tilde{\phi}^1, \dots, \tilde{\phi}^{j-1}, \phi^j, \tilde{\phi}^{j+1}, \dots \tilde{\phi}^A)^T$$

is a bounded, compactly supported scaling vector with accuracy $m \geq 1$ that generates the same MRA as Φ.

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- Build $M(z)$ to be an upper triangular matrix, with ones on the main diagonal and $\sum_{k\in\Lambda_i} c_{ik} z^k$ in the i, j position ($i < j$). Here $z = e^{-i\omega}$.

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- Build $M(z)$ to be an upper triangular matrix, with ones on the main diagonal and $\sum_{k\in\Lambda_i} c_{ik} z^k$ in the i, j position ($i < j$). Here $z = e^{-i\omega}$.
- The vector that works is

$$\tilde{\Phi} = \mathcal{F}^{-1}\left(M(z)\hat{\Phi}(z)\right)$$

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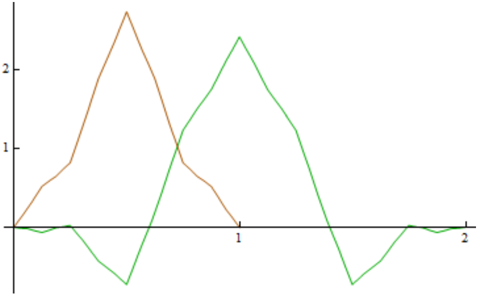
Example. Recall the DGHM example.

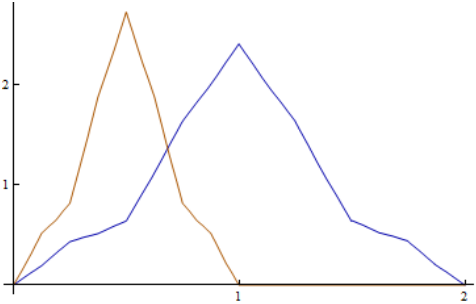


- $\phi^1(t) \geq 0$.
- supp $\left(\phi^1\right) = [0,1]$ and supp $\left(\phi^2\right) = [0,2]$.
- ϕ^1, ϕ^2 are continuous and have polynomial accuracy $m = 2$.
- the partition of unity coefficients are

$$c_1 = (1+\sqrt{2})^{-1} \quad \text{and} \quad c_2 = \sqrt{2}c_1.$$

We take the index set $\Lambda_2 = \{0, 1\}$ with $c_{20} = c_{21} = \frac{1}{2}$ so that

$$\tilde{\phi}^2 = \phi^2(t) + \frac{1}{2}\phi^1(t) + \frac{1}{2}\phi^1(t-1)$$



The new partition of unity coefficients are

$$d_1 = c_1 > 0$$
$$d_2 = c_2 - c_1\left(c_{20} + c_{21}\right) > 0$$

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Example. G. Plonka and V. Strela used a two-scale similarity transform in the frequency domain to construct the scaling vector

$$\Phi(t) = \sum_{k=0}^{2} C_k \cdot \Phi(2t - k)$$

where

$$C_0 = \frac{1}{20}\begin{bmatrix} -7 & 15 \\ -4 & 10 \end{bmatrix}, C_1 = \frac{1}{20}\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}, C_2 = \frac{1}{20}\begin{bmatrix} -7 & -15 \\ 4 & 10 \end{bmatrix}$$

- Φ and its translates are not orthogonal.
- Φ is continuous and has approximation order $m = 3$.
- $\text{supp}\left(\phi^1\right) = \text{supp}\left(\phi^2\right) = [0, 2]$.
- Partition of unity: $c_1 = 0$ and $c_2 = 1$.

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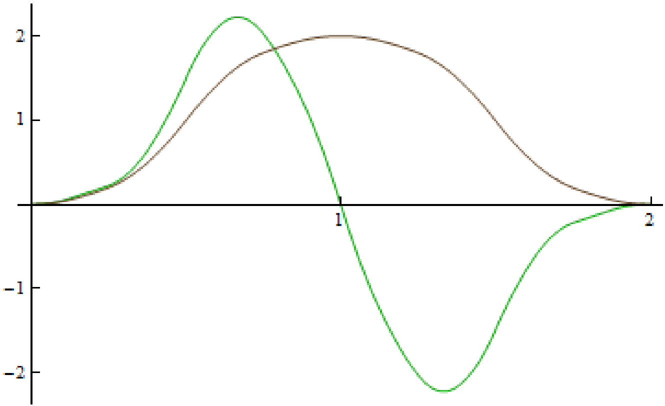
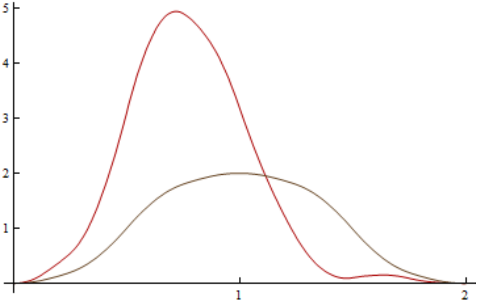

2
1
−1
−2
1
2

We take the index set $\Lambda_1 = \{0\}$ with $c_{10} = 1.6$ so that

$$\tilde{\phi}^1 = \phi^1(t) + 1.6\phi^2(t)$$



The new partition of unity coefficients are

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- Φ generates an MRA for $L^2(\mathbb{R})$.

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- For $\ell = 1, \ldots, A$, suppose that the set S of non-zero functions

$$S = \left\{ \overline{\phi}_k^{\ell}(t) \right\}_{k \in \mathbb{Z}}$$

where

$$\overline{\phi}_k^{\ell}(t) = \phi^{\ell}(t-k)\Big|_{[0,\infty)}$$

are linearly independent and let $n(S)$ be the number of elements in S.

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are linearly independent and let $n(S)$ be the number of elements in S.
- S is simply the right shifts of ϕ^{ℓ} and the left shifts for $k = 1, \ldots, M_{\ell} - 1$. Here, $[0, M_{\ell}]$ is the support of ϕ^{ℓ}.

We define the left edge functions $\phi_{L,n}$ by

$$\phi_{L,n}(t) = \sum_{\ell=1}^{A} \sum_{k=1-M_\ell}^{0} \mathbf{f}_{nk}^{\ell} \overline{\phi}_k^{\ell}(t)$$

We have

$$\phi_{L,n}(t) = t^n \quad \text{on} \quad [0,1]$$

We are building the edge function by simply taking those $\overline{\phi}_k^{\ell}(t)$ that contribute to t^n on $[0,1]$.

Proposition. The $\phi_{L,n}(t)$ satisfy a matrix refinement equation.

The proof is straightforward and you end up with

$$\phi^{L,n}(t) = 2^{-n}\phi_{L,n}(2t) + \sum_{j=2-2M_n}^{N} \mathbf{q}_{nj}\Phi(2t-j)$$

for each $n = 0, 1, \ldots, m-1$, where

$$\mathbf{q}_{nj} = \begin{cases} \sum_{k=1-M_n}^{0} \mathbf{f}_{nk} C_{j-2k} - 2^{-n}\mathbf{f}_{nj}, & j \in \{1-M_n, \ldots, 0\} \\ \sum_{k=1-M_n}^{0} \mathbf{f}_{nk} C_{j-2k}, & j \in \{2-2M_n, \ldots, -M_n\} \cup \{1, \ldots, N\} \end{cases}$$

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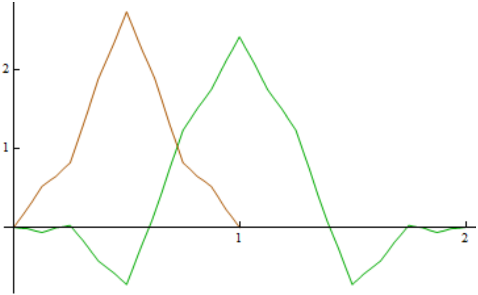
Example. Recall the DGHM example.



- $\phi^1(t) \geq 0$, $M_1 = 1$, $M_2 = 2$.
- ϕ^1, ϕ^2 are continuous and have polynomial accuracy $m = 2$.
- the partition of unity coefficients are

$$\mathbf{f}_{00} = \left(\frac{1}{1+\sqrt{2}}, \frac{\sqrt{2}}{1+\sqrt{2}} \right)^T .$$

- $S = \left\{ \phi^1(t), \overline{\phi}^2(t), \overline{\phi}^2(t+1) \right\}$.

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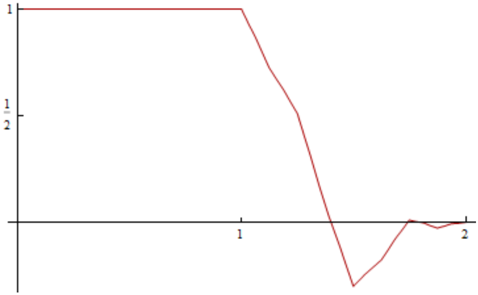
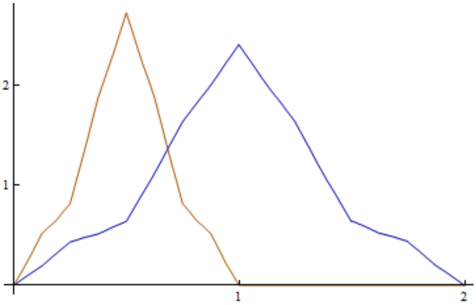
We can easily write down the formula for the edge function (noting that $f^{\ell}_{0,k} = f^{\ell}_{0,0}$ for all $k \in \mathbb{Z}$):

$$\phi_{L,0}(t) = f^1_{0,0}\phi^1(t) + f^2_{0,0}\left(\overline{\phi}^2(t) + \overline{\phi}^2(t+1)\right)$$



If we want a nonnegative edge function, then we need to use the scaling vector:



Here,

$$\mathbf{f}_{00} = (c_1, c_2 - c_1)^T = \left(\frac{1}{1+\sqrt{2}}, 3 - 2\sqrt{2} \right)^T$$

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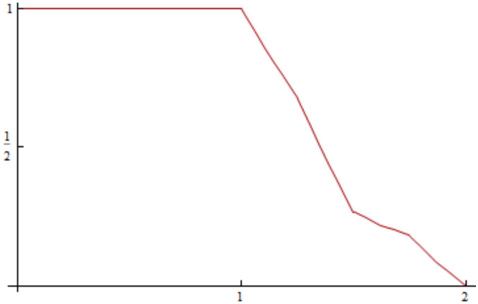
The resulting edge function:



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- $M_1 = M_2 = 2$.
- Partition of unity: $\mathbf{f}_{0,0} = (0, 1)^T$.
- We also need $\mathbf{f}_{1,0} = \left(\frac{1}{6}, 1\right)^T$.

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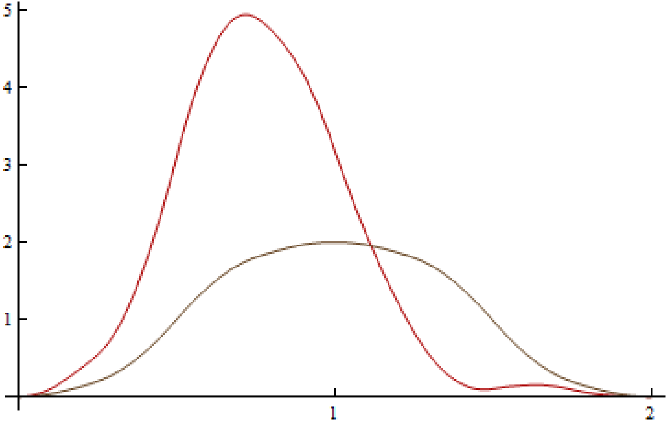
We use the scaling vector



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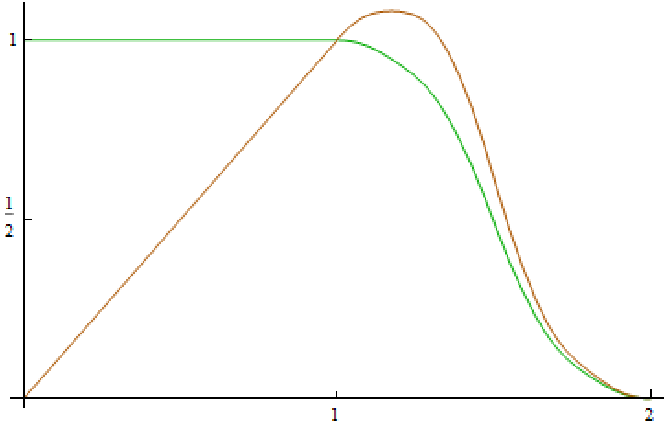
We can write down the (nonnegative!) edge functions:

$$\phi_{L,0}(t) = \overline{\phi}^1(t) + \overline{\phi}^1(t+1)$$

$$\phi_{L,1}(t) = \overline{\phi}^1(t) - \frac{1}{6}\left(\overline{\phi}^2(t) + \overline{\phi}^2(t+1)\right)$$



Theorem. (**D. Ruch**, **PVF**) For some index set B, let $\{L_i\}$ be a finite set of left edge functions with support $[0, \delta_i]$ and assume that $\{L_i, \phi^\ell(\cdot - k)\}_{i,\ell,k\geq 0}$ is a linearly independent set. Then $\{L_i(2^j\cdot), \phi^\ell(2^j\cdot - k)\}_{i,\ell,k\geq 0}$ is a Riesz basis for V_j, where $L^2[0,\infty) = \overline{\cup_j V_j}$.

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- In both cases it seems we are missing an edge function - $\phi_{L,1}(t)$ for DGHM and $\phi_{L,2}(t)$ for Plonka/Strela.
- But it turns out that we don't need them - in both cases we were able to find constants α_j so that

$$\sum_{j=0}^{m-2} \alpha_j \phi_{L,j}(t) + \sum_{j=1}^{2} \alpha_j + m - 2\overline{\phi}^j(t-k) = t^{m-1}$$

on $[0, 1]$. (Here, $m = 2$ for DGHM and $m = 3$ for Plonka/Strela).

- It is natural to ask if this holds in a more general setting.

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- **Important Note:** We are assuming the total support of the scaling vector, $M_1 + \cdots + M_A = m + 1$. All our example scaling vectors plus the Daubechies family of scaling functions satisfy this property.

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- We assume $M_1 + \cdots M_A = m + 1$.

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- Then we seek $\alpha_0, \ldots, \alpha_m$ such that

$$\sum_{j=0}^{m-A} \alpha_j \phi_{L,j}(t) + \sum_{\ell=1}^{A} \alpha_{\ell+m-A} \phi^j(t) = t^{m-1}$$

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on $[0, 1]$.

- Rewriting using the linearly independent $\overline{\phi}_k^\ell(t)$ and the definition of the edge functions gives the following system:

$$\sum_{j=0}^{m-A} \alpha_j \left[\sum_{\ell=1}^{A} \sum_{k=1-M_\ell}^{0} f_{j,k}^\ell \overline{\phi}_k^\ell(t) \right] + \sum_{\ell=1}^{A} \alpha_{\ell+m-A} \overline{\phi}_0^\ell(t) = \sum_{\ell=1}^{A} \sum_{k=1-M_\ell}^{0} f_{m-1,k}^\ell \overline{\phi}_k^\ell(t)$$

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- This system can be rewritten in the form $M\alpha = \mathbf{b}$ where

$$M = \begin{bmatrix} Q & 0 \\ R & I \end{bmatrix}$$

Here, M has dimension $(m+1) \times (m+1)$, I is the $A \times A$ identity matrix, and Q is an $(m-A+1) \times (m-A+1)$ matrix.

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- Certainly if Q is nonsingular, our assertion holds.

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We can further refine Q. If we set

$$E_k^\ell = \begin{bmatrix} f_{0,k}^\ell & f_{1,k}^\ell & f_{2,k}^\ell & \cdots & f_{m-2,k}^\ell \end{bmatrix}$$

then we can write Q as

$$Q = \begin{bmatrix} E_1^1 \\ \vdots \\ E_{M_1-1}^1 \\ \hline \vdots \\ \hline E_1^A \\ \vdots \\ E_{M_A-1}^A \end{bmatrix}$$

- A lemma due to G. Strang says that

$$f^{\ell}_{j,k+1} = \sum_{i=0}^{j} \binom{j}{i} f^{\ell}_{i,j}$$

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$$f_{j,k+1}^{\ell} = \sum_{i=0}^{j} \binom{j}{i} f_{i,j}^{\ell}$$

- We can reformulate this lemma in terms of our rows E_k^{ℓ} and the upper triangular Pascal matrix

$$P_U = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & \cdots \\ 0 & 0 & 1 & 3 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ & & \vdots & & \end{bmatrix}$$

- In terms of the Pascal matrix, Strang's lemma says

$$E^{\ell}_{k+1} = E^{\ell}_{k} P_U = \cdots = E^{\ell}_{1} P_U^{k-1} = E^{\ell}_{0} P_U^{k}$$

or

$$E^{\ell}_{k+1} P_U^{-k} = E^{\ell}_{k} P_U^{-k+1} = \cdots = E^{\ell}_{1} P_U^{-1} = E^{\ell}_{0}$$

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or

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- Then Q becomes

$$Q = \begin{bmatrix} E_0^1 P_U^{-1} \\ \vdots \\ E_0^1 P_U^{1-M_1} \\ \hline \vdots \\ \hline E_0^A P_U^{-1} \\ \vdots \\ E_0^A P_U^{1-M_1} \end{bmatrix}$$

- It is not clear that Q is always nonsingular. It would seem there needs to be conditions placed on the E_0^ℓ.

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- In the case where $M_1 = L > 1$ and $M_k = 1$, $k = 2, \ldots, A$, (DGHM, for example) we can reduce Q to

$$Q = \begin{bmatrix} E_0^1 P_U^{-1} \\ E_0^1 P_U^{-2} \\ \vdots \\ E_0^1 P_U^{1-L} \end{bmatrix}$$

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Using the lower triangular Pascal matrix ($P_L = P_U^T$), we can write $Q =$

$$\begin{bmatrix} E_0^1 P_U^{-1} \\ E_0^1 P_U^{-2} \\ \vdots \\ E_0^1 P_U^{1-L} \end{bmatrix} = \begin{bmatrix} E_0^1 \\ E_0^1 P_U^{-1} \\ \vdots \\ E_0^1 P_U^{2-L} \end{bmatrix} P_U^{-1} = P_L \left(P_L^{-1} \begin{bmatrix} E_0^1 \\ E_0^1 P_U^{-1} \\ \vdots \\ E_0^1 P_U^{2-L} \end{bmatrix} P_U^{-1} \right)$$

$$= P_L \left(\begin{bmatrix} E_0^1 \\ E_0^1 \left(P_U^{-1} - I \right) \\ E_0^1 \left(P_U^{-1} - I \right)^2 \\ \vdots \\ E_0^1 \left(P_U^{-1} - I \right)^{L-2} \end{bmatrix} P_U^{-1} \right) = L \cdot U$$

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- The first matrix is lower triangular with determinant 1 while the second matrix is upper triangular with determinant $Cf^1_{0,0}$, $C > 0$. So if $f^1_{0,0} \neq 0$, our matrix Q is nonsingular in this case.

- The first matrix is lower triangular with determinant 1 while the second matrix is upper triangular with determinant $Cf^1_{0,0}$, $C > 0$. So if $f^1_{0,0} \neq 0$, our matrix Q is nonsingular in this case.
- This is certainly the case for the DGHM scaling vector as $f^1_{0,0} = \left(1 + \sqrt{2}\right)^{-1}$.

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Thank You - Questions?