Sampling and Recovery of Sparse Signals and its Application to Image Feature Extraction

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New Trends and Directions in Harmonic Analysis, Fractional Operator Theory, and Image Analysis

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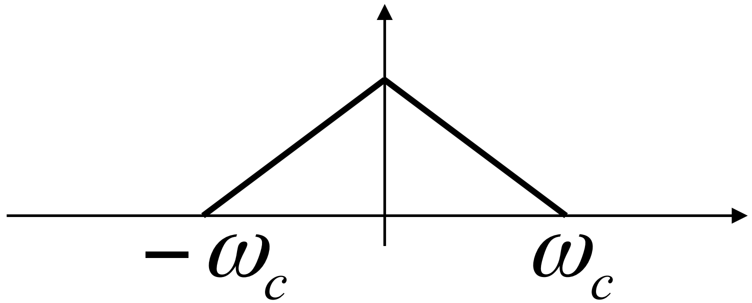
Yamaguchi Prefecture

Classical Sampling Theorem

- Whittaker (1915), Kotelnikov (1933), Someya (1948), and Shannon (1948)

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\omega t} dt$$

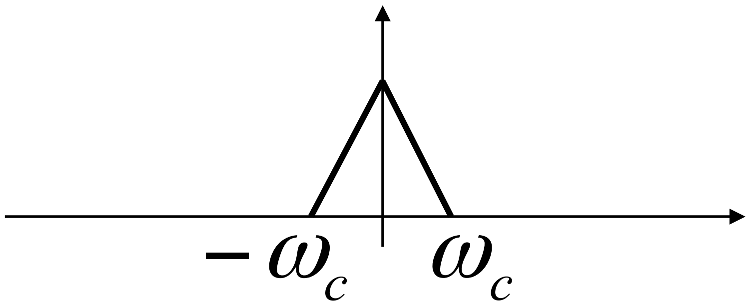


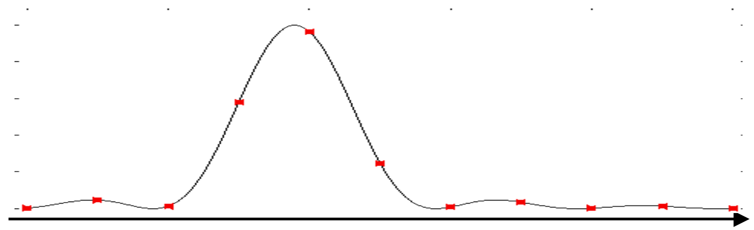
$$\omega_s \geq 2\omega_c$$

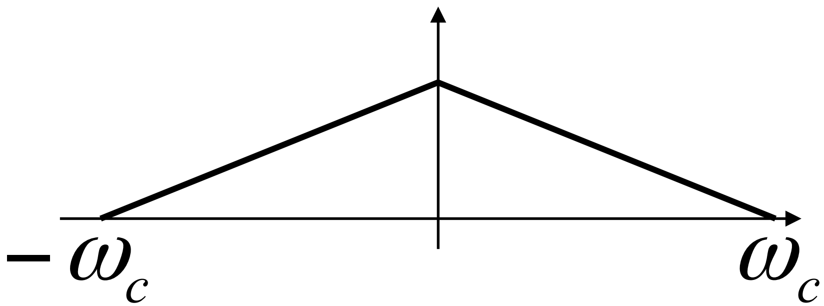
$$f(t) = \frac{2\omega_c}{\omega_s} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\omega_s}\right) \frac{\sin 2\pi\omega_c (t - k/\omega_s)}{2\pi\omega_c (t - k/\omega_s)}$$

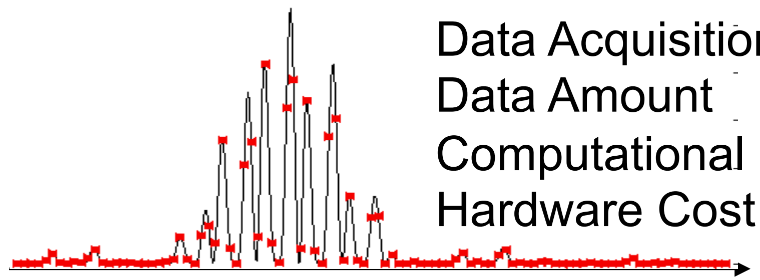
Nyquist Interval

Fourier Transform





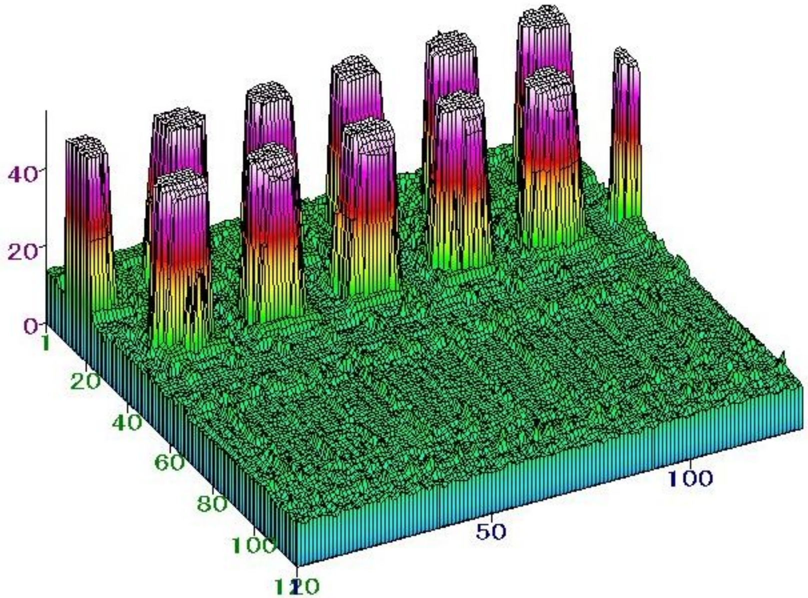




Data Acquisition
Data Amount
Computational Cost
Hardware Cost

Surface Profiling by WLI

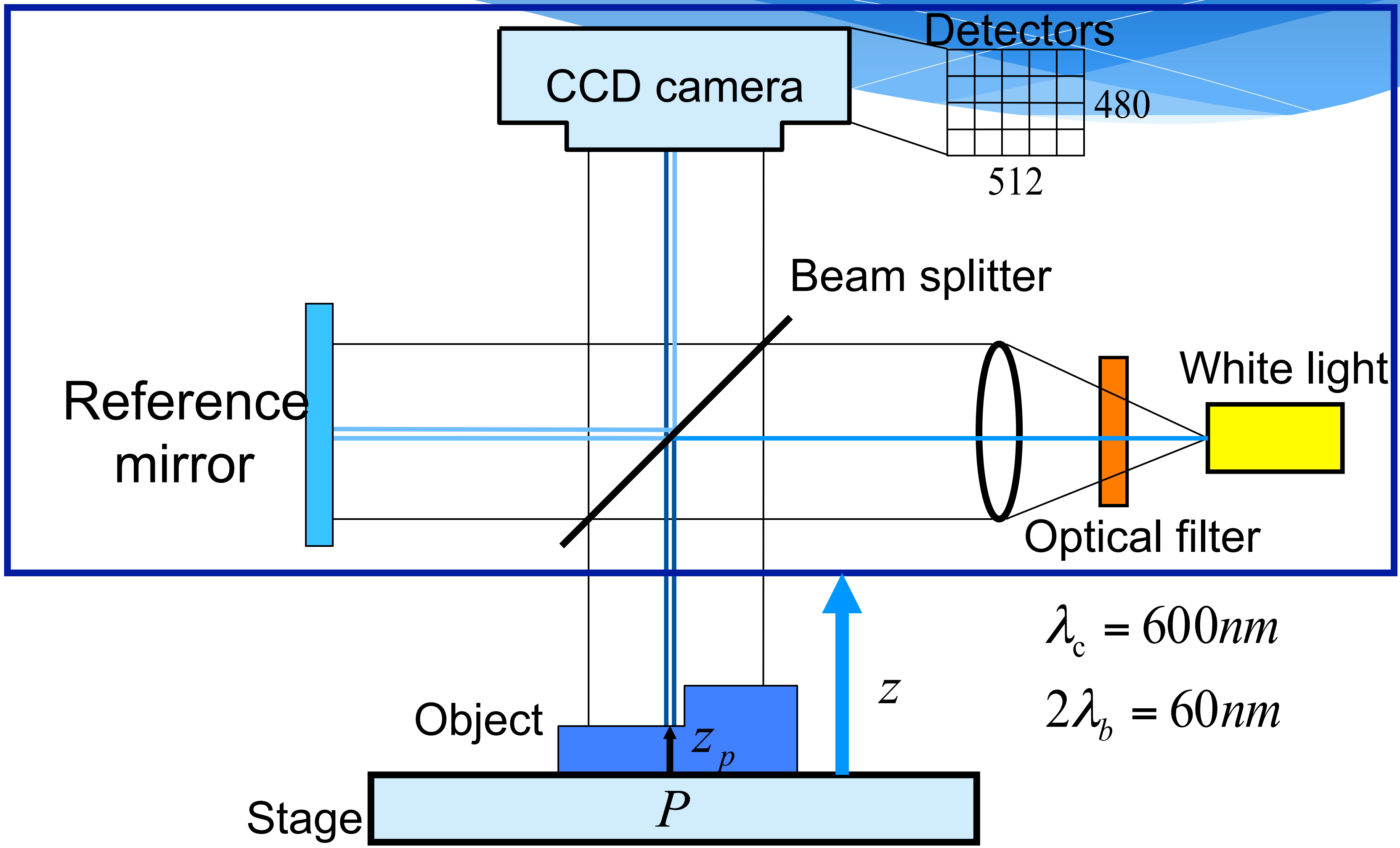
WLI (White-Light Interferometry):
Technique for surface profiling of semiconductors, LCD, Plastic films, etc…



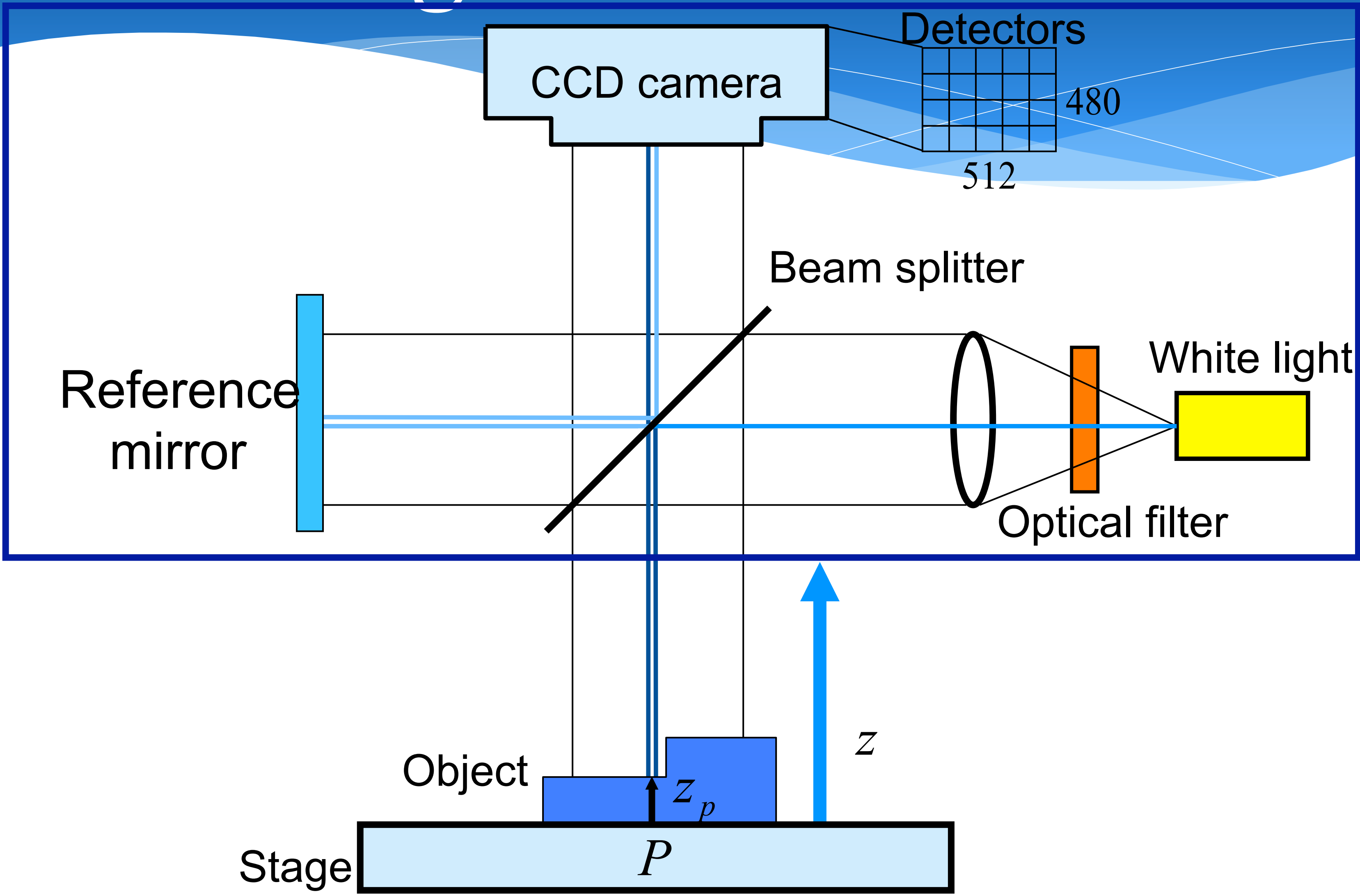
$$1[\text{pixel}] = 5.9[\mu m] \times 5.9[\mu m]$$

http://www.scn.tv/user/torayins/SP-500.html

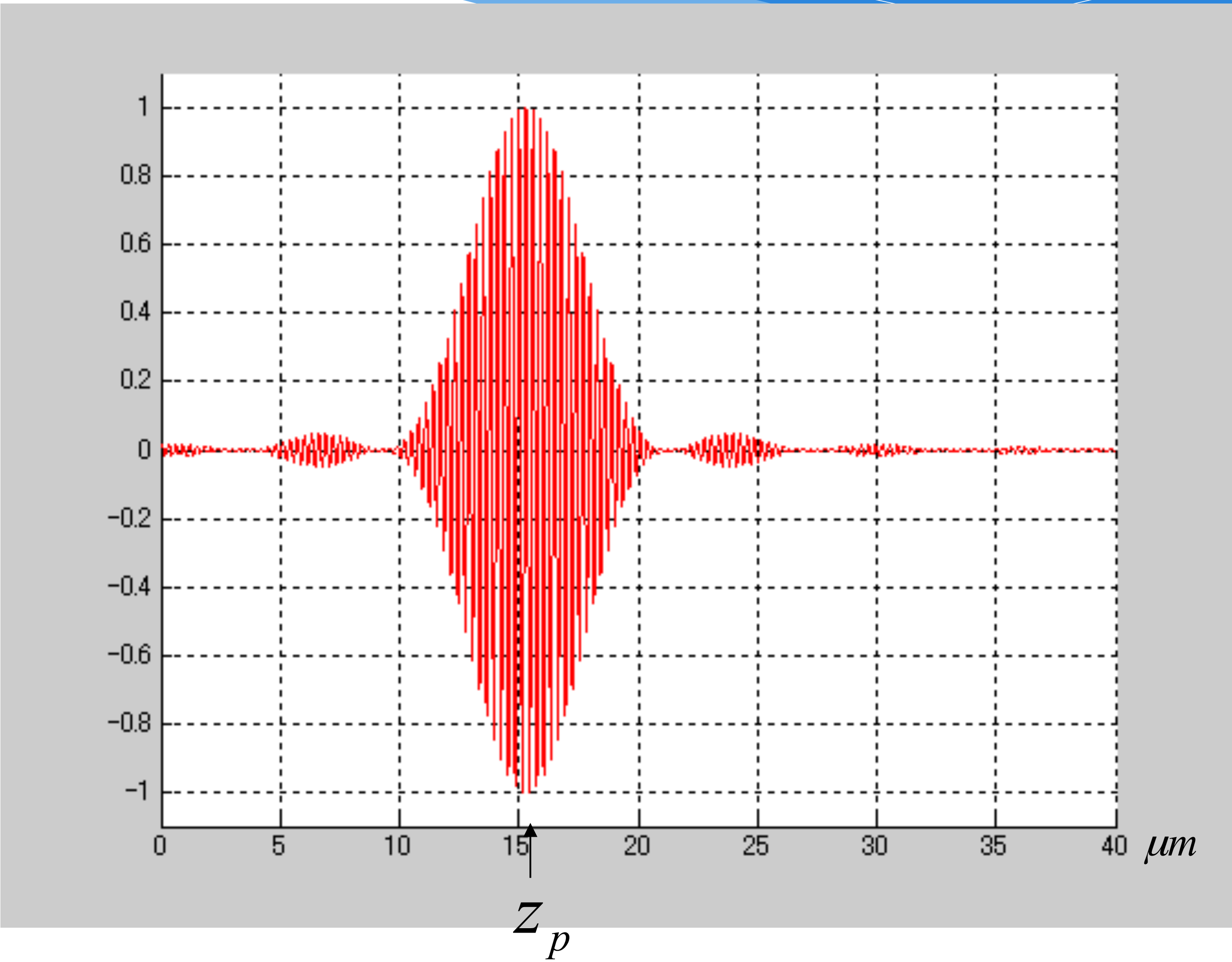
White-Light Interferometer



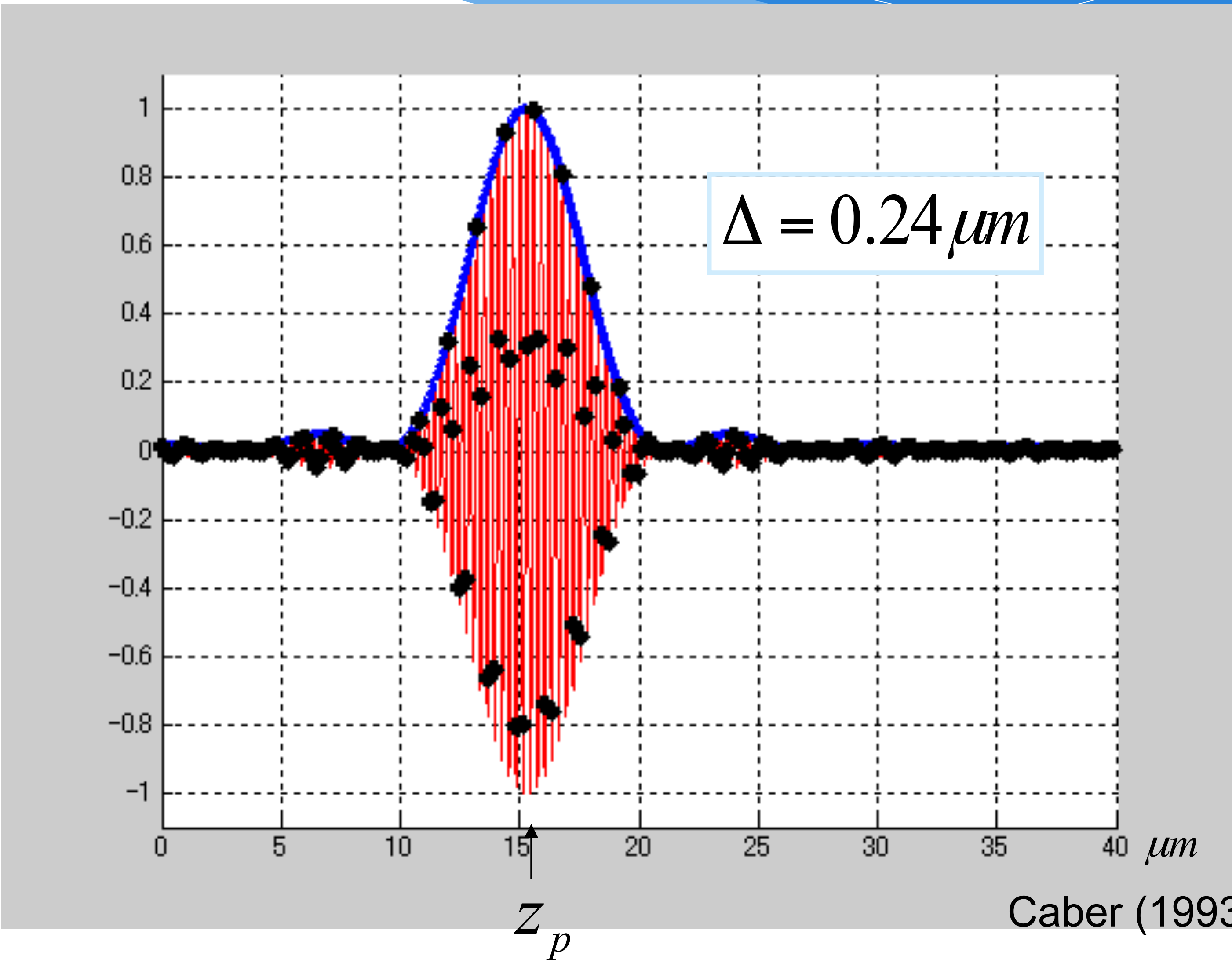
White-Light Interferometer



White-Light Interferogram

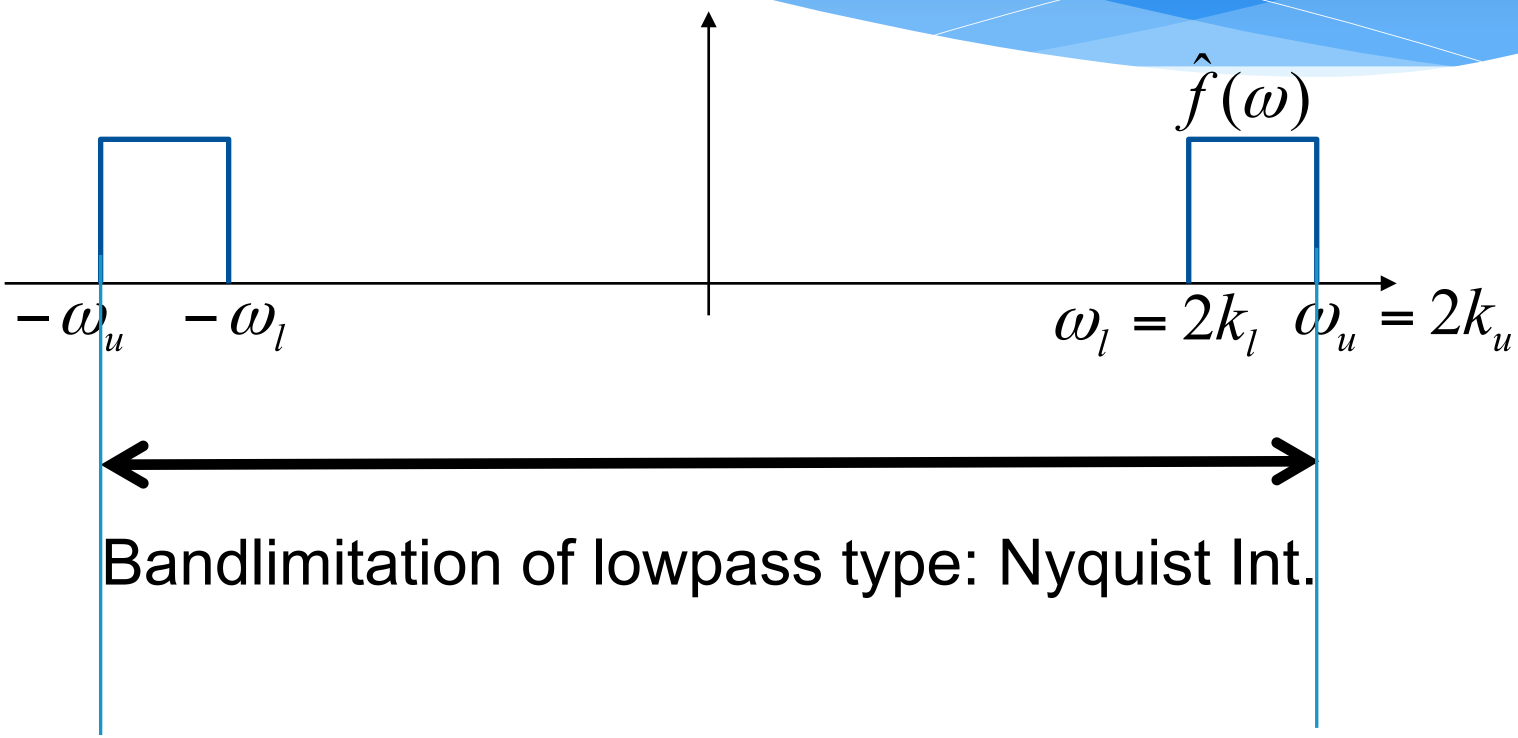


Nyquist Sampling for WLI

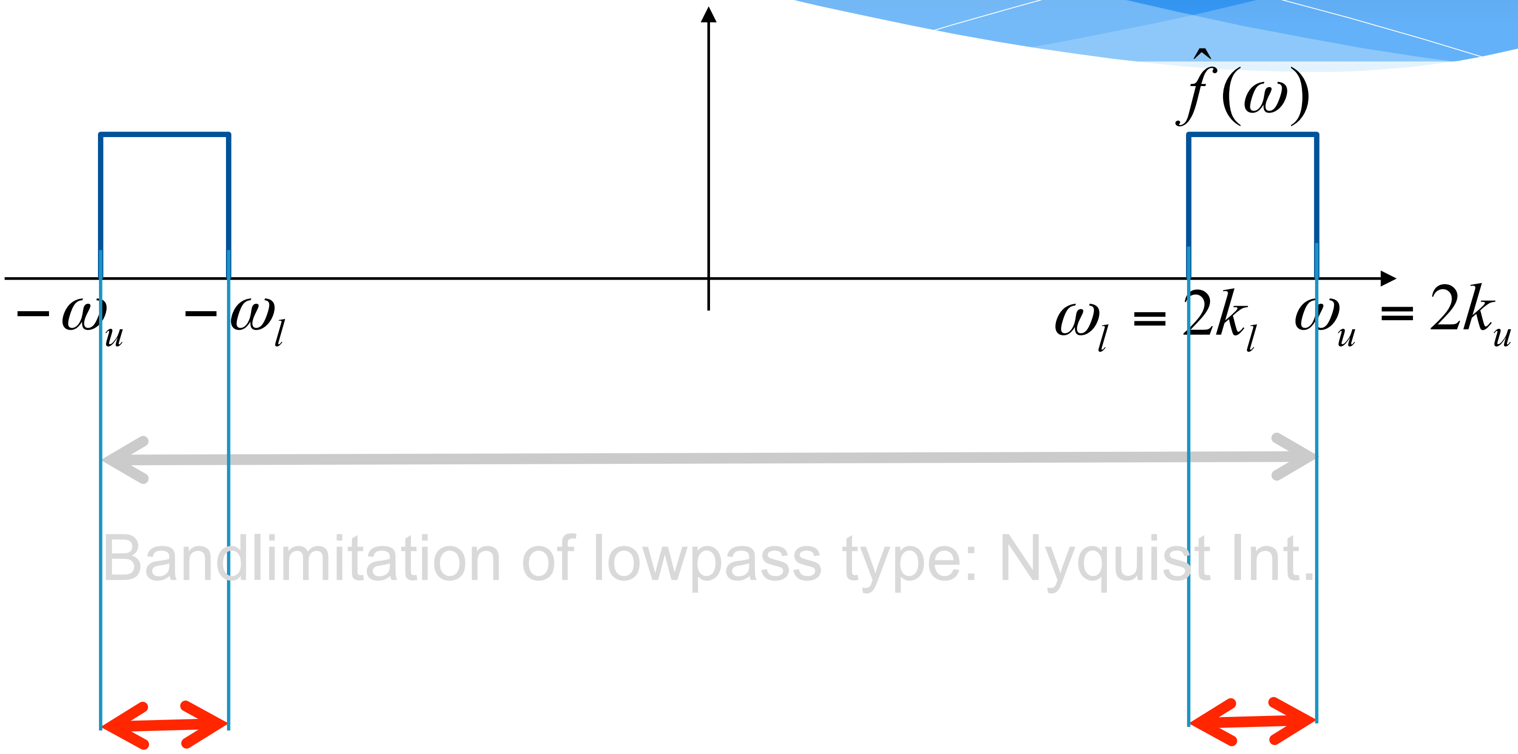


Caber (1993)

Bandlimitation of WLI

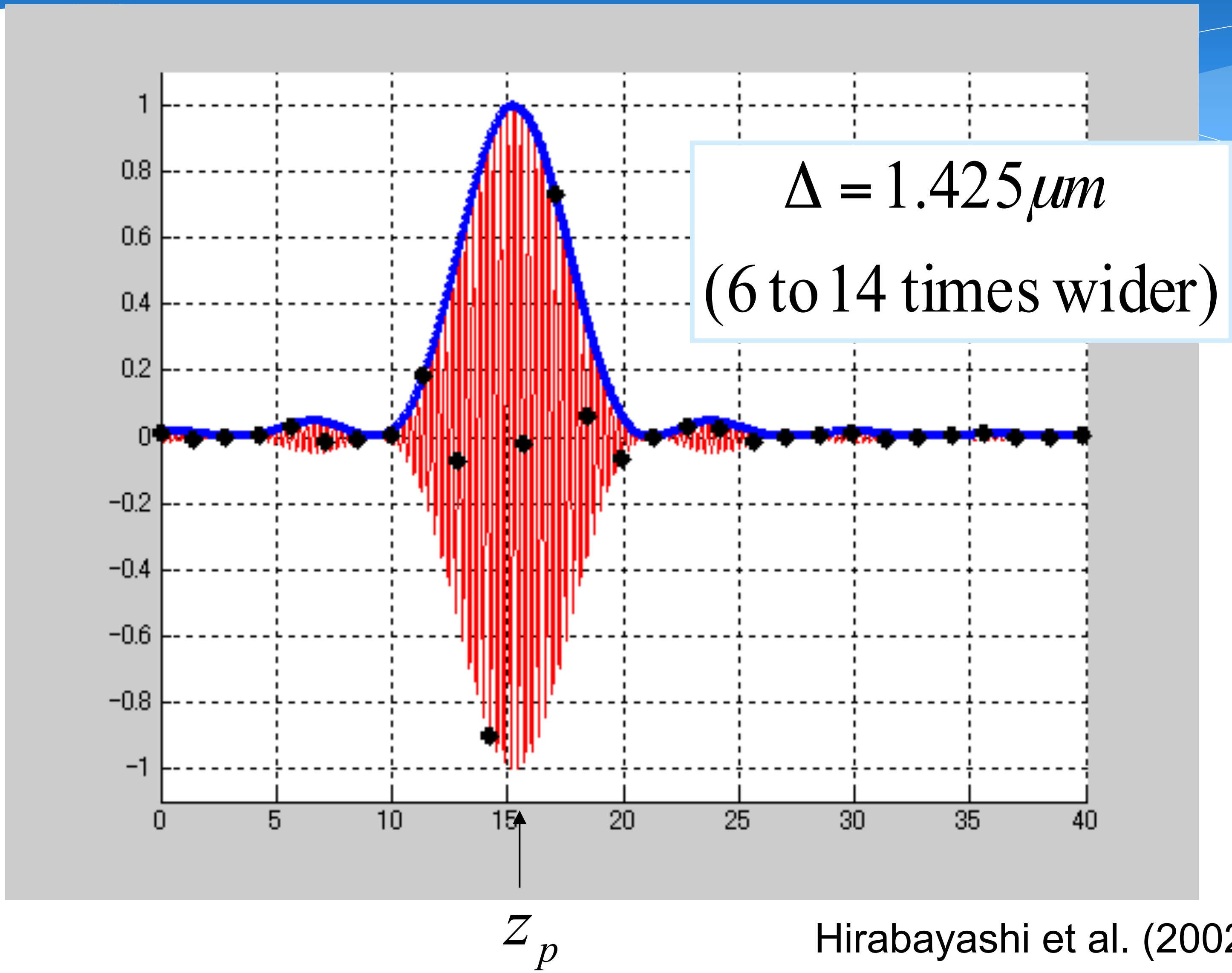


Bandlimitation of WLI



Bandlimitation of Bandpass Type⇒Kohlenberg (1953)

Interval of Our Algorithm



Hirabayashi et al. (2002)

Surface Profiler SP500

Toray Engineering, Co. Ltd.

http://www.scn.tv/user/torayins/SP-500.html

New Class of Signals

IEEE TRANSACTIONS ON SIGNAL PROCESSING, VOL. 50, NO. 6, JUNE 2002 1417

Sampling Signals With Finite Rate of Innovation

Martin Vetterli, *Fellow, IEEE*, Pina Marziliano, and Thierry Blu, *Member, IEEE*

***Abstract*—Consider classes of signals that have a finite number of degrees of freedom per unit of time and call this number the rate of innovation. Examples of signals with a finite rate of innovation include streams of Diracs (e.g., the Poisson process), nonuniform splines, and piecewise polynomials.**

Even though these signals are not bandlimited, we show that they can be sampled uniformly at (or above) the rate of innovation using an appropriate kernel and then be perfectly reconstructed. Thus, we prove sampling theorems for classes of signals and kernels that generalize the classic "bandlimited and sinc kernel" case. In particular, we show how to sample and reconstruct periodic and finite-length streams of Diracs, nonuniform splines, and piecewise polynomials using sinc and Gaussian kernels. For infinite-length signals with finite local rate of innovation, we show local sampling and reconstruction based on spline kernels.

The key in all constructions is to identify the innovative part of a signal (e.g., time instants and weights of Diracs) using an annihilating or locator filter: a device well known in spectral analysis and error-correction coding. This leads to standard computational procedures for solving the sampling problem, which we show through experimental results.

Applications of these new sampling results can be found in signal processing, communications systems, and biological systems.

***Index Terms*—Analog-to-digital conversion, annihilating filters, generalized sampling, nonbandlimited signals, nonuniform splines, piecewise polynomials, poisson processes, sampling.**

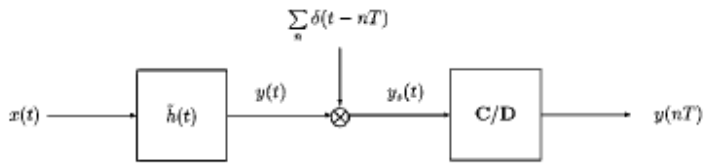


Fig. 1. Sampling setup: $x(t)$ is the continuous-time signal; $\tilde{h}(t) = h(-t)$ is the smoothing kernel; $y(t)$ is the filtered signal; T is the sampling interval; $y_s(t)$ is the sampled version of $y(t)$; and $y(nT), n \in \mathbb{Z}$ are the sample values. The box C/D stands for continuous-to-discrete transformation and corresponds to reading out the sample values $y(nT)$ from $y_s(t)$.

The intermediate signal $y_s(t)$ corresponding to an idealized sampling is given by

$$y_s(t) = \sum_{n \in \mathbb{Z}} y(nT)\,\delta(t - nT). \qquad (2)$$

This setup is shown in Fig. 1.

When no smoothing kernel is used, we simply have $y(nT) = x(nT)$, which is equivalent to (1) with $h(t) = \delta(t)$. This simple model for having access to the continuous-time world is typical for acquisition devices in many areas of science and technology, including scientific measurements, medical and biological signal processing, and analog-to-digital converters.

Outline

- Introduction of new class of signals
  - As an extension of bandlimited signals
- Sampling and Reconstruction
  - Noiseless case
  - Noisy case
- Application
  - Compression of ECG signals
  - Line-edge extraction

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Extension of Classical Samp. Th.

$$f(t) = \frac{2\omega_c}{\omega_s} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\omega_s}\right) \frac{\sin 2\pi\omega_c (t - k/\omega_s)}{2\pi\omega_c (t - k/\omega_s)}$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k s(t - k\Delta t)$$

$s(t)$: given function with FT $\hat{s}(\omega)$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k s(t - t_k)$$

Rate of Innovation

Vetterli et al. (2002)

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\, s(t - t_k) \qquad s(t):\ \text{given function}$$

Unknown parameters: (t_k, c_k)

$C_f(t_a, t_b) =$ number of $t_k \in [t_a, t_b]$ & corresponding c_k

Rate of innovation: $\rho = \lim_{\tau \to \infty} \frac{1}{\tau} C_f(-\tau/2, \tau/2)$

If $\rho < \infty$, $f(t)$ is called

Signals with Finite Rate of Innovation

More General Case

Vetterli et al. (2002)

$$f(t) = \sum_{k=-\infty}^{\infty} \sum_{r=0}^{R-1} c_{k,r}\, s_r(t - t_k) \qquad s_r(t):\ \text{given function}$$

$$\text{Unknown parameters: } (t_k, c_{k,r})$$

$$C_f(t_a, t_b) = \text{number of } t_k \in [t_a, t_b] \text{ \& corresponding } c_{k,r}$$

$$\text{Rate of innovation: } \rho = \lim_{\tau \to \infty} \frac{1}{\tau} C_f(-\tau/2, \tau/2)$$

If $\rho < \infty$, $f(t)$ is called

Signals with Finite Rate of Innovation

Local Rate of Innovation

For a fixed τ, a local rate of innovation at time t is defined by

$$\rho(t) = \frac{1}{\tau} C_f\left(t - \tau/2, t + \tau/2\right).$$

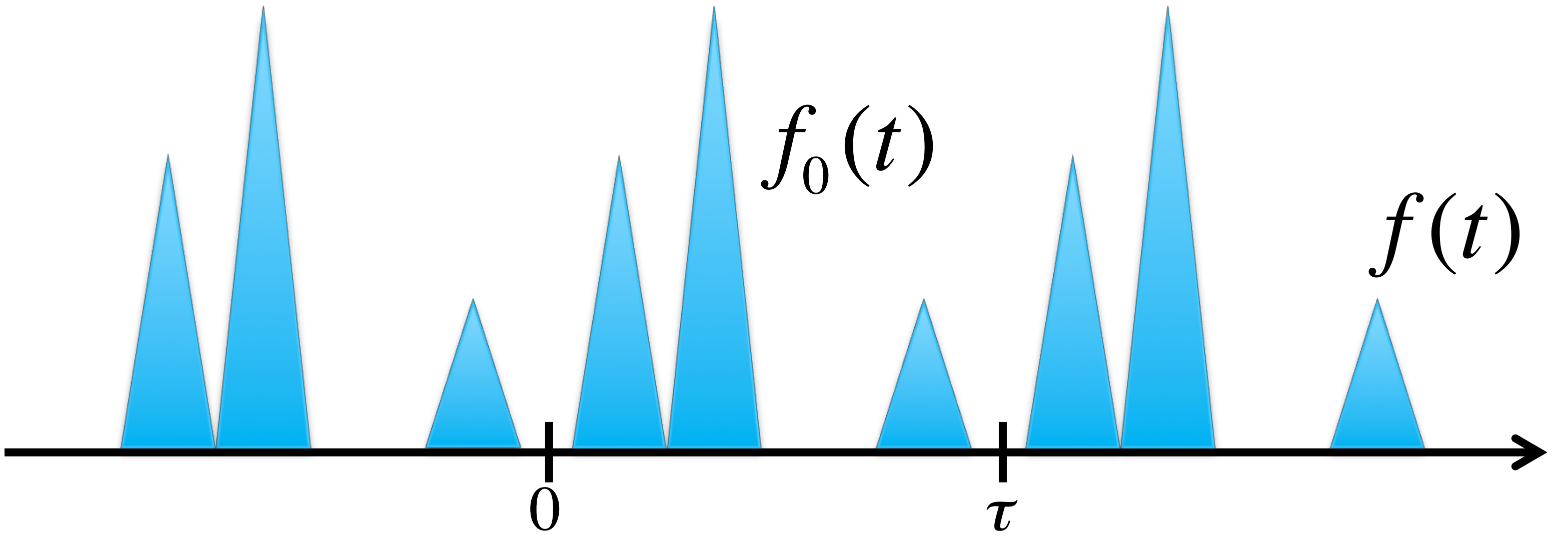
Then, a local rate of innovation is defined by

$$\rho = \max_t \rho(t).$$

Periodic Signals with FRI

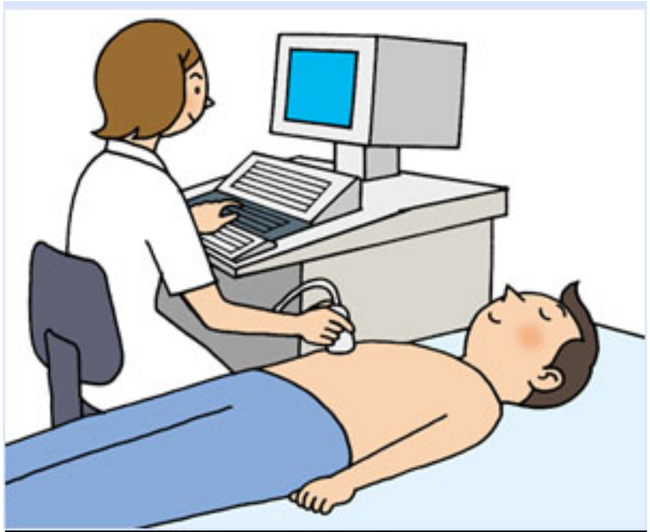
(Vetterli *et al.*,2002)

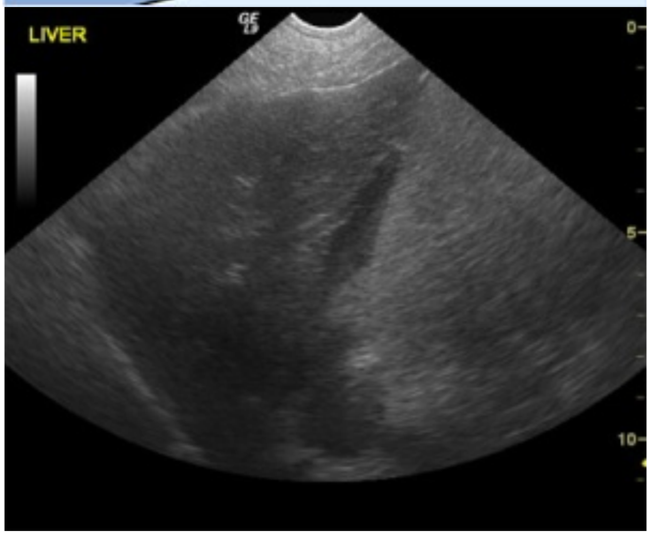
$$f_0(t) = \sum_{k=0}^{K-1} c_k s(t - t_k) \quad (0 \le t_0 < t_1 < \cdots < t_{K-1} < \tau)$$

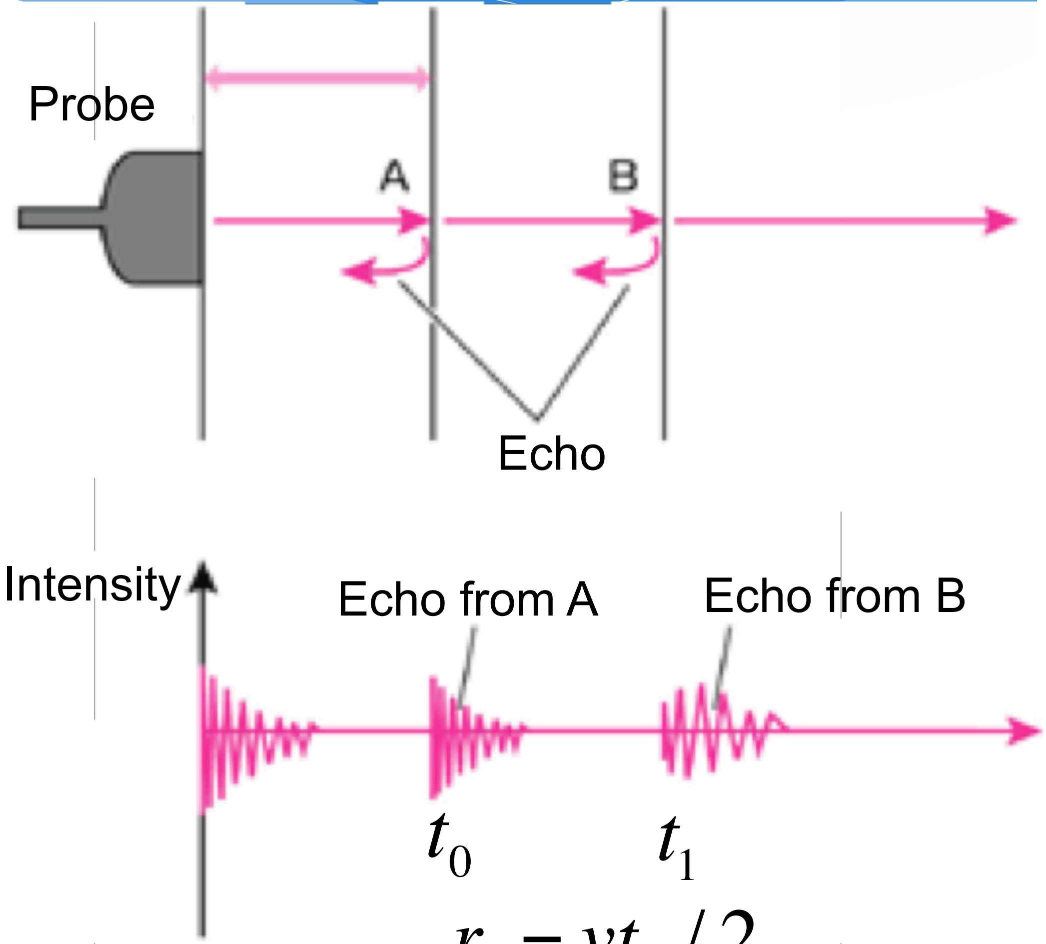


Rate of innovation: $\rho = \frac{2K}{\tau}$

Echo Imaging



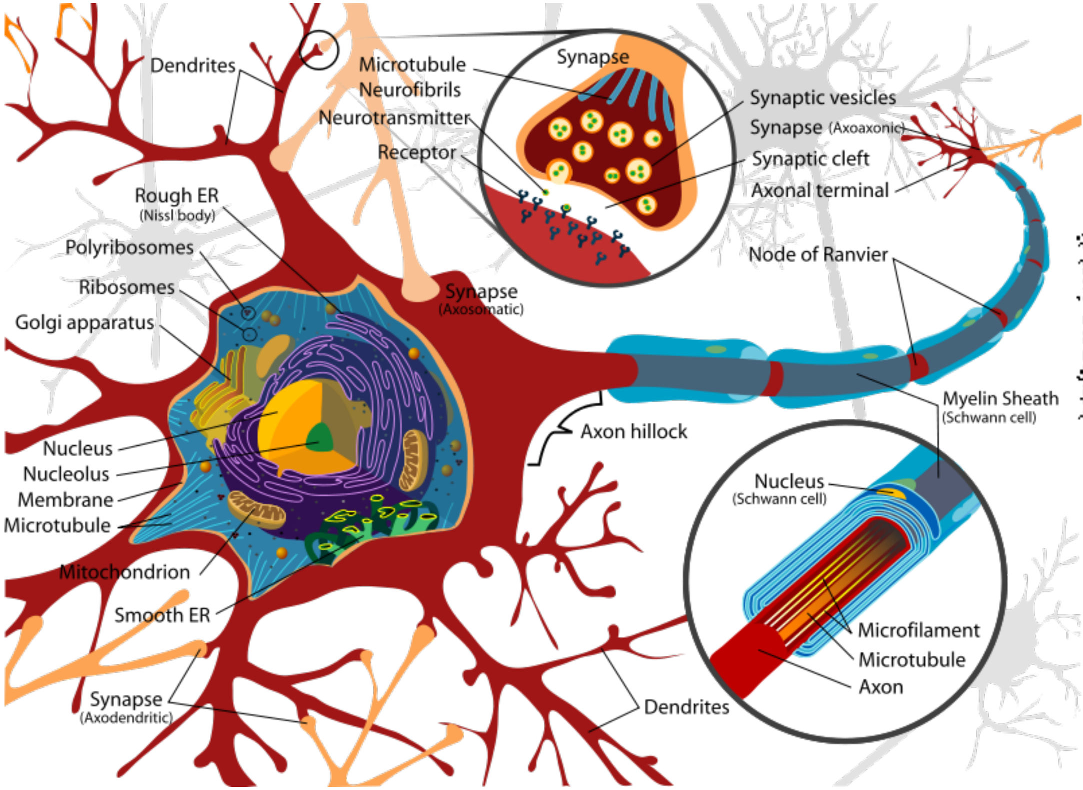


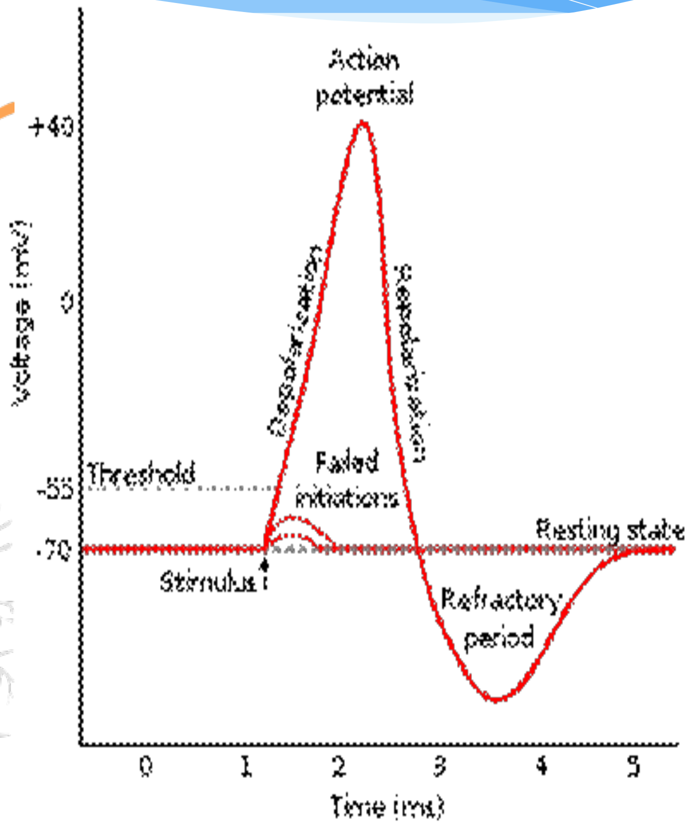


$$r_k = vt_k / 2$$

$$v = 1,530 m / s$$

Neuron Pulses





Stream of Diracs

The most important signal with FRI is

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \delta(t - t_k),$$

$$\text{where } \int_{-\infty}^{\infty} \delta(t - t_k)\phi(t)\,dt = \phi(t_k).$$

This is because the convolution generates

$$g(t) = (s * f)(t) = \sum_{k=-\infty}^{\infty} c_k s(t - t_k).$$

$$\hat{g}(\omega) = \hat{s}(\omega)\hat{f}(\omega)$$

Stream of Derivative of Diracs

$$f(t) = \sum_{k=-\infty}^{\infty} \sum_{r=0}^{R-1} c_{k,r}\, \delta^{(r)}(t - t_k)$$

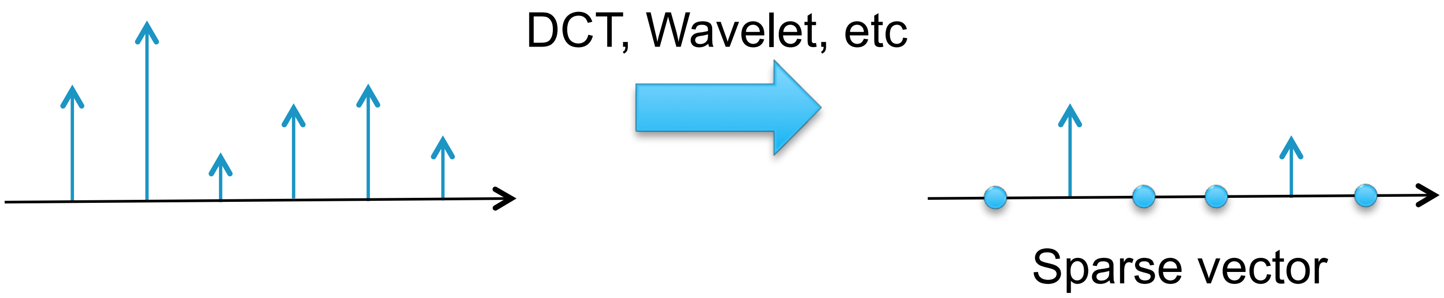
$$\int_{-\infty}^{\infty} \delta^{(r)}(t - t_k)\, \phi(t)\, dt = (-1)^r \phi^{(r)}(t_k)$$

$$g(t) = (s * f)(t) = \sum_{k=-\infty}^{\infty} \sum_{r=0}^{R-1} (-1)^r c_{k,r}\, s^{(r)}(t - t_k)$$

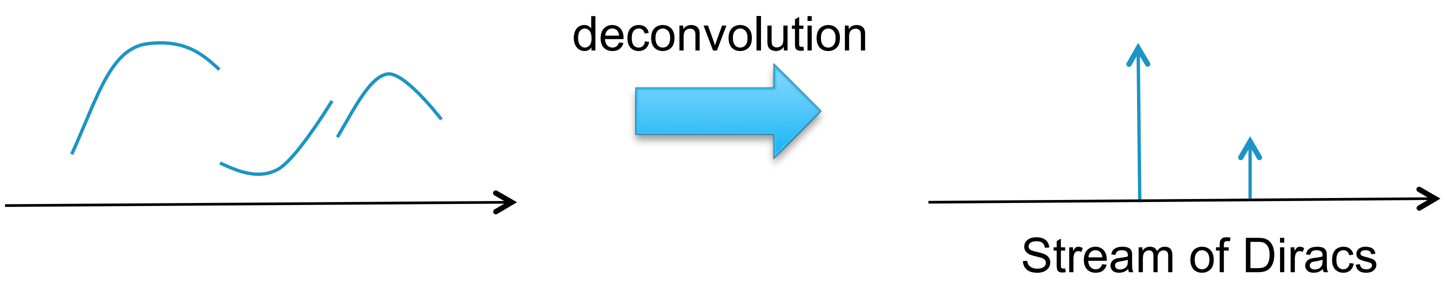
: special case of $f(t) = \sum_{k=-\infty}^{\infty} \sum_{r=0}^{R-1} c_{k,r}\, s_r(t - t_k)$ with $s_r(t) = s^{(r)}(t)$.

Two Types of Sparsity

- Discrete case (Compressed sensing):



- Continuous case (FRI theory):



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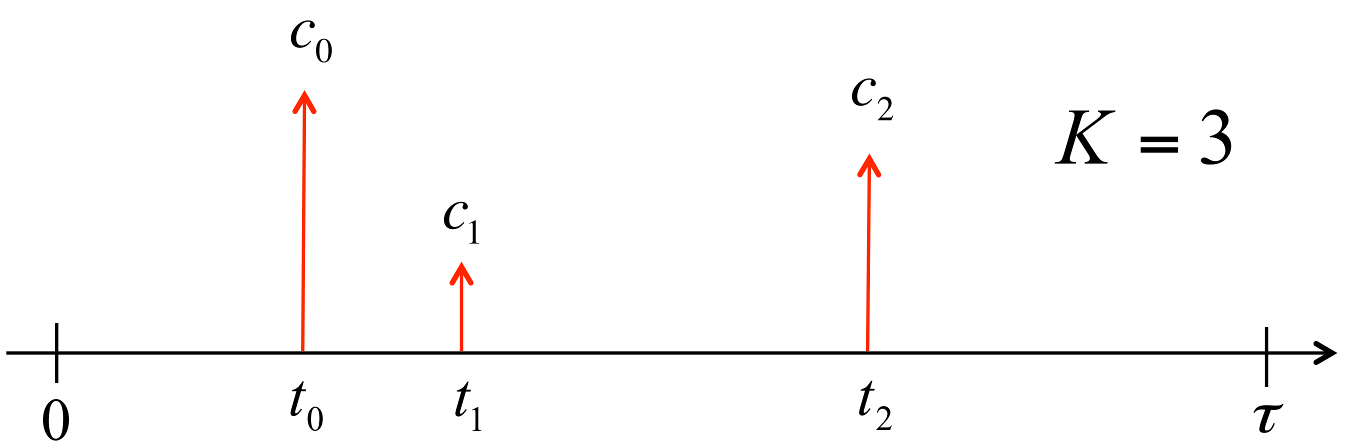
Periodic Stream of Diracs

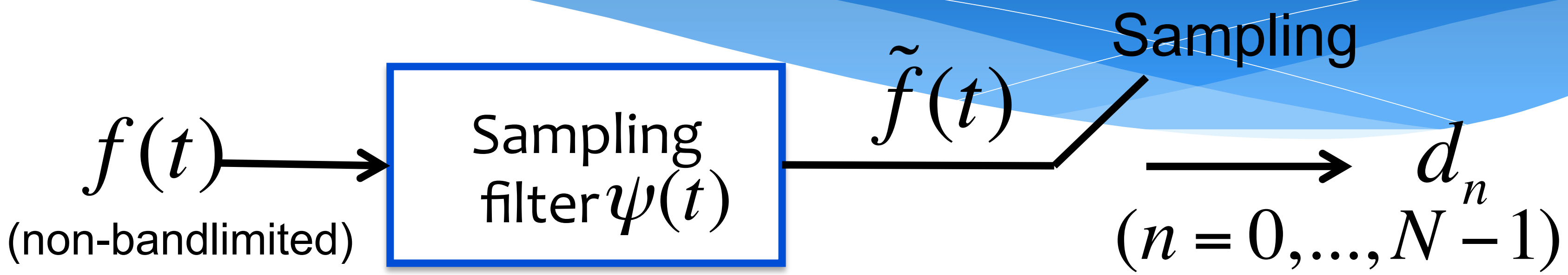
$$f_0(t) = \sum_{k=0}^{K-1} c_k \delta(t - t_k)$$

Given: τ, K Unknown: t_k, c_k



Rate of Innovation $\rho = 2K / \tau$

Sampling Filter



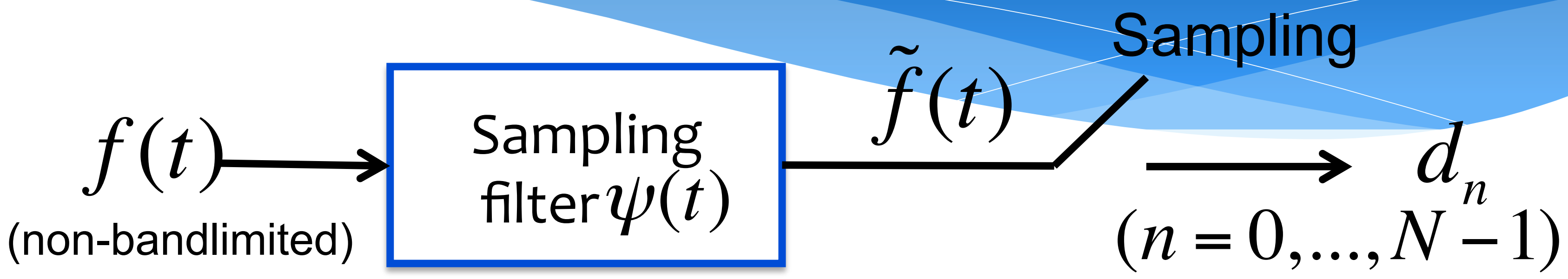
$$d_n = \langle f(t), \psi(t-nT) \rangle = \int_{-\infty}^{\infty} f(t)\overline{\psi(t-nT)}\,dt$$

$$T = \tau / N$$

Proposed sampling filters

	Support	Number of pulse
Sinc (Vetterli et al., 2002)	Infinite	> 10
Spline (Dragotti et al., 2007)	Finite	<10
Sum of Sinc (Tur et al., 2011)	Finite	>10

Sinc Sampling Filter



$$d_n = \langle f(t), \psi(t - nT) \rangle = \int_{-\infty}^{\infty} f(t)\overline{\psi(t - nT)}\,dt$$

$$T = \tau / N$$

$$\psi(t) = B\mathrm{sinc}(Bt), \text{ where } B \geq \rho = \frac{2K}{\tau}$$

Sinc Samples

$$d_n = \int_{-\infty}^{\infty} f(t)\overline{\psi(t-nT)}\,dt$$

$$= \int_{-\infty}^{\infty} \left\{ \sum_{k'=-\infty}^{\infty} f_0(t-k'\tau) \right\} B\mathrm{sinc}(t-nT)\,dt$$

$$= \int_{-\infty}^{\infty} \left\{ \sum_{k'=-\infty}^{\infty} f_0(t) B\mathrm{sinc}(t-nT+k'\tau) \right\} dt$$

Poisson Sum Form.

$$= \int_{0}^{\tau} f_0(t) \left\{ \frac{1}{\tau} \sum_{p=-P}^{P} \exp \frac{-i2p\pi(t-nT)}{\tau} \right\} dt \qquad \left(P = \left\lfloor \frac{B\tau}{2} \right\rfloor \leq \frac{B\tau}{2} \right)$$

$$= \sum_{p=-P}^{P} \underbrace{\left\{ \frac{1}{\tau} \int_{0}^{\tau} f_0(t) \exp \frac{-i2p\pi t}{\tau}\,dt \right\}}_{\text{Fourier coefficient of } f(t)} \exp \frac{i2pn\pi}{N}$$

Sinc Samples vs. Fourier Coef.

$$d_n = \sum_{p=-P}^{P} \hat{d}_p \exp\left(\frac{i2pn\pi}{N}\right)$$

DFT

$$\hat{d}_p = \frac{1}{N} \sum_{n=0}^{N-1} d_n \exp\left(-\frac{i2pn\pi}{N}\right)$$

$$N \geq 2P + 1$$

Fourier Coefficients

$$\hat{d}_p = \frac{1}{\tau}\int_0^{\tau} f_0(t)\exp\frac{-i2p\pi t}{\tau}dt$$

$$= \frac{1}{\tau}\int_0^{\tau}\left\{\sum_{k=0}^{K-1} c_k\delta(t-t_k)\right\}\exp\frac{-i2p\pi t}{\tau}dt$$

$$= \frac{1}{\tau}\sum_{k=0}^{K-1} c_k\exp\frac{-i2p\pi t_k}{\tau}$$

$$= \frac{1}{\tau}\sum_{k=0}^{K-1} c_k u_k^p \qquad u_k = \exp\frac{-i2p\pi t_k}{\tau}$$

Sinc Sampling

$$d_n \longrightarrow \boxed{\text{DFT}} \longrightarrow \hat{d}_p = \sum_{k=0}^{K-1} c_k u_k^p \qquad (u_k = e^{-i2\pi t_k/\tau})$$

Cf) Spectral Estimation, Direction of Arrival (DoA)

Problem	FRI theory	Spectral	DoA
Parameters	Time delay	Frequency	Direction
K	# of pulse	# of component	# of object
Sampling	?	Nyquist	Nyquist

Annihilation in case of K=1

$$(u_0 = e^{-i2\pi t_0/\tau})$$

Sequence of Fourier Coef.

$$\hat{d}_{-P} = c_0 u_0^{-P}$$
$$\hat{d}_{-P+1} = c_0 u_0^{-P+1}$$
$$\vdots$$
$$\hat{d}_0 = c_0$$
$$\vdots$$
$$\hat{d}_{P-1} = c_0 u_0^{P-1}$$
$$\hat{d}_P = c_0 u_0^{P}$$

Filter:

$$a = [a_0, a_1] = [1, -u_0]$$

Convolution:

$$\left(a * \hat{d}\right)_p = \sum_{q=0}^{1} a_q \hat{d}_{p-q}$$
$$= a_0 \hat{d}_p + a_1 \hat{d}_{p-1}$$
$$= c_0 u_0^p + (-u_0) c_0 u_0^{p-1}$$
$$= 0$$

Annihilation in case of K=2

$$(u_k = e^{-i2\pi t_k/\tau})$$

Sequence of Fourier Coef.

$$\hat{d}_{-P} = c_0 u_0^{-P} + c_1 u_1^{-P}$$

$$\hat{d}_{-P+1} = c_0 u_0^{-P+1} + c_1 u_1^{-P+1}$$

$$\vdots$$

$$\hat{d}_0 = c_0 + c_1$$

$$\vdots$$

$$\hat{d}_{P-1} = c_0 u_0^{P-1} + c_1 u_1^{P-1}$$

$$\hat{d}_P = c_0 u_0^{P} + c_1 u_1^{P}$$

Filter:

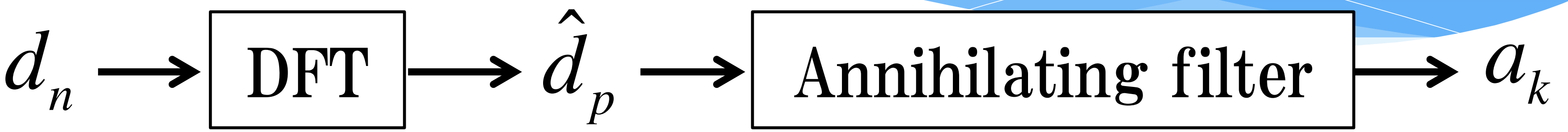
$$\begin{aligned} a &= [a_0, a_1, a_2] \\ &= [1, -(u_0 + u_1), u_0 u_1] \\ &= [1, -u_0] * [1, -u_1] \end{aligned}$$

Convolution:

$$\begin{aligned} \left(a * \hat{d}\right)_p &= a_0 \hat{d}_p + a_1 \hat{d}_{p-1} + a_2 \hat{d}_{p-2} \\ &= c_0 u_0^p (1 - u_0 z^{-1})(1 - u_1 z^{-1})\Big|_{z=u_0} \\ &\quad + c_1 u_1^p (1 - u_0 z^{-1})(1 - u_1 z^{-1})\Big|_{z=u_1} \\ &= 0 \end{aligned}$$

Annihilating Filter

(Vetterli *et al.*,2002)



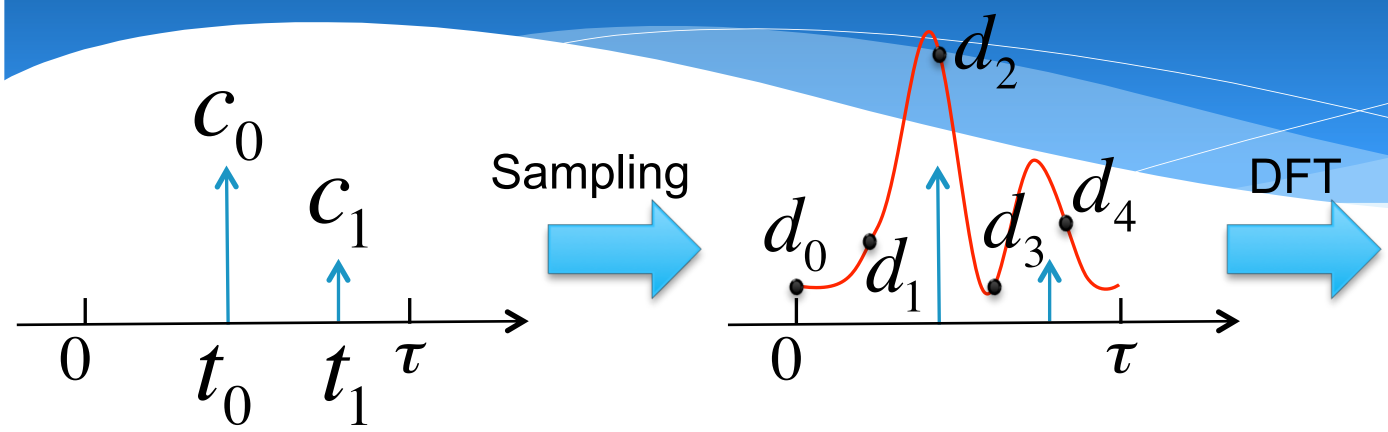
$$\hat{d}_p + a_1 \hat{d}_{p-1} + \ldots + a_K \hat{d}_{p-K} = 0 \;\; (p = 0, 1, \ldots, K-1)$$

: Annihilating relation

$$1 + a_1 z^{-1} + \ldots + a_K z^{-K} = \prod_{k=0}^{K-1} (1 - u_k z^{-1})$$

$$u_k = e^{-i 2\pi t_k / \tau}$$

In Case of K=2



$$\hat{d}_{-2}, \hat{d}_{-1}, \cdots, \hat{d}_2$$

Annihilation

$$\begin{cases} \hat{d}_0 + a_1 \hat{d}_{-1} + a_2 \hat{d}_{-2} = 0 \\ \hat{d}_1 + a_1 \hat{d}_0 + a_2 \hat{d}_{-1} = 0 \\ \hat{d}_2 + a_1 \hat{d}_1 + a_2 \hat{d}_0 = 0 \end{cases}$$

$$1 + a_1 z^{-1} + a_2 z^{-2} = (1 - u_0 z^{-1})(1 - u_1 z^{-1}) = 0$$

$$t_k = -\frac{\tau \angle (u_k)}{2\pi}$$

$$\begin{cases} \hat{d}_{-2} = c_0 u_0^{-2} + c_1 u_1^{-2} \\ \vdots \\ \hat{d}_2 = c_0 u_0^2 + c_1 u_1^2 \end{cases}$$

$$c_k$$

Th. 1 Stream of Diracs

(Vetterli et al., 2002)

Assume that B in $\psi(t) = B\mathrm{sinc}(Bt)$ satisfies

$$B \geq \frac{2K}{\tau} \, (= \rho)$$

and that

$$N \geq 2P + 1$$

with $P = \lfloor B\tau/2 \rfloor$. Then, the sinc kernel samples $\{d_n\}_{n=0}^{N-1}$ are a sufficient characterization of the τ-periodic stream of Diracs.

Sampling Rate

$$B \geq \frac{2K}{\tau} \quad\Rightarrow\quad K \leq \frac{B\tau}{2}$$

$$P = \left\lfloor \frac{B\tau}{2} \right\rfloor \quad\Rightarrow\quad P \leq \frac{B\tau}{2} < P + 1$$

$$K \leq P$$

$$N \geq 2P + 1 \geq 2K + 1$$

* Sampling rate for this scheme

$$\omega_s \equiv \frac{N}{\tau} \geq \frac{2K+1}{\tau} > \frac{2K}{\tau} = \rho$$

Periodic Derivative of Diracs

$$f_0(t) = \sum_{k=0}^{K-1} \sum_{r=0}^{R-1} c_{k,r}\, \delta^{(r)}(t - t_k)$$

Degree of freedom in a period:

K from time instants, and KR from coef.

Rate of innovation:

$$\rho = \frac{K + KR}{\tau} = \frac{K(R+1)}{\tau}$$

Fourier Coefficients

$$\hat{d}_p = \frac{1}{\tau}\int_0^\tau f_0(t)\exp\frac{-i2p\pi t}{\tau}dt$$

$$= \frac{1}{\tau}\int_0^\tau \left\{\sum_{k=0}^{K-1}\sum_{r=0}^{R-1} c_{k,r}\delta^{(r)}(t-t_k)\right\}\exp\frac{-i2p\pi t}{\tau}dt$$

$$= \frac{1}{\tau}\sum_{k=0}^{K-1}\sum_{r=0}^{R-1} c_{k,r}\left(\frac{i2p\pi}{\tau}\right)^r \underbrace{\exp\frac{-i2p\pi t_k}{\tau}}_{u_k^p}$$

$$= \sum_{k=0}^{K-1}\sum_{r=0}^{R-1} \tilde{c}_{k,r}p^r u_k^p \qquad \tilde{c}_{k,r} = \frac{1}{\tau}\left(\frac{i2\pi}{\tau}\right)^r c_{k,r}$$

Annihilation in Case of K=1 & R=2

$$(u_0 = e^{-i2\pi t_0/\tau})$$

Sequence of Fourier Coef.

$$\hat{d}_{-P} = \tilde{c}_{0,0}u_0^{-P} + \tilde{c}_{0,1}(-P)u_0^{-P}$$

$$\hat{d}_{-P+1} = \tilde{c}_{0,0}u_0^{-P+1} + \tilde{c}_{0,1}(-P+1)u_0^{-P+1}$$

$$\vdots$$

$$\hat{d}_0 = \tilde{c}_{0,0}$$

$$\vdots$$

$$\hat{d}_{P-1} = \tilde{c}_{0,0}u_0^{P-1} + \tilde{c}_{0,1}(P-1)u_0^{P-1}$$

$$\hat{d}_P = \tilde{c}_{0,0}u_0^{P} + \tilde{c}_{0,1}(P)u_0^{P}$$

Filter:

$$a = [a_0, a_1, a_2] = [1, -u_0] * [1, -u_0] = [1, -2u_0, u_0^2]$$

Convolution:

$$(a * \hat{d})_p = 0$$

Annihilation in General Case

Sequence of Fourier Coef.

$$\hat{d}_p = \sum_{k=0}^{K-1} \sum_{r=0}^{R-1} \tilde{c}_{k,r} p^r u_k^p$$

Convolution:

$$(a * \hat{d})_p = 0$$

Filter: $(u_k = e^{-i2\pi t_k/\tau})$

$$a = [a_0, a_1, \ldots, a_{KR}]$$

$$= \underbrace{[1, -u_0] * \ldots * [1, -u_0]}_{R \text{ times}}$$

$$* \underbrace{[1, -u_1] * \ldots * [1, -u_1]}_{R \text{ times}}$$

$$\vdots$$

$$* \underbrace{[1, -u_{K-1}] * \ldots * [1, -u_{K-1}]}_{R \text{ times}}$$

Th. 2 Derivative of Diracs

(Hirabayashi, 2012)

Assume that B in $\psi(t) = B\mathrm{sinc}(Bt)$ satisfies

$$B \geq \frac{2KR}{\tau} \left(> \rho = \frac{K(R+1)}{\tau} \right)$$

and that

$$N \geq 2P + 1$$

with $P = \lfloor B\tau/2 \rfloor$. Then, the sinc kernel samples $\{d_n\}_{n=0}^{N-1}$ are a sufficient characterization of the τ-periodic stream of differentiated Diracs.

Original Statement in 2002

Theorem 3: Consider a periodic stream of differentiated Diracs $x(t)$ with period τ, as in (32). Take as a sampling kernel $h_B(t) = B\,\mathrm{sinc}(Bt)$, where B is greater or equal to the rate of innovation ρ given by (33), and sample $(h_B * x)(t)$ at N uniform locations $t = nT, n = 0, \ldots, N-1$, where $N \geq 2M+1$ and $M = \lfloor B\tau/2 \rfloor$. Then, the samples

$$y_n = \langle h_B(t - nT), x(t) \rangle, \quad n = 0, \ldots, N-1 \qquad (37)$$

are a sufficient characterization of $x(t)$.

$$\rho = \frac{K + \tilde{K}}{\tau}. \qquad (33)$$

Derivative of General Pulses

$$g_0(t) = \sum_{k=0}^{K-1} \sum_{r=0}^{R_k-1} c_{k,r} s^{(r)}(t - t_k),$$

Since

$$g(t) = \sum_{k'=-\infty}^{\infty} g_0(t - k'\tau) = (s * f)(t),$$

where $f(t)$ is the stream of derivative of Diracs,

$$\hat{d}_p(g) = \hat{s}\left(\frac{2p\pi}{\tau}\right) \hat{d}_p(f)$$

Th. 3 Derivative of General Pulses

Assume that B in $\psi(t) = B\text{sinc}(Bt)$ satisfies

$$B \geq \frac{2KR}{\tau} \left(> \rho = \frac{K(R+1)}{\tau} \right)$$

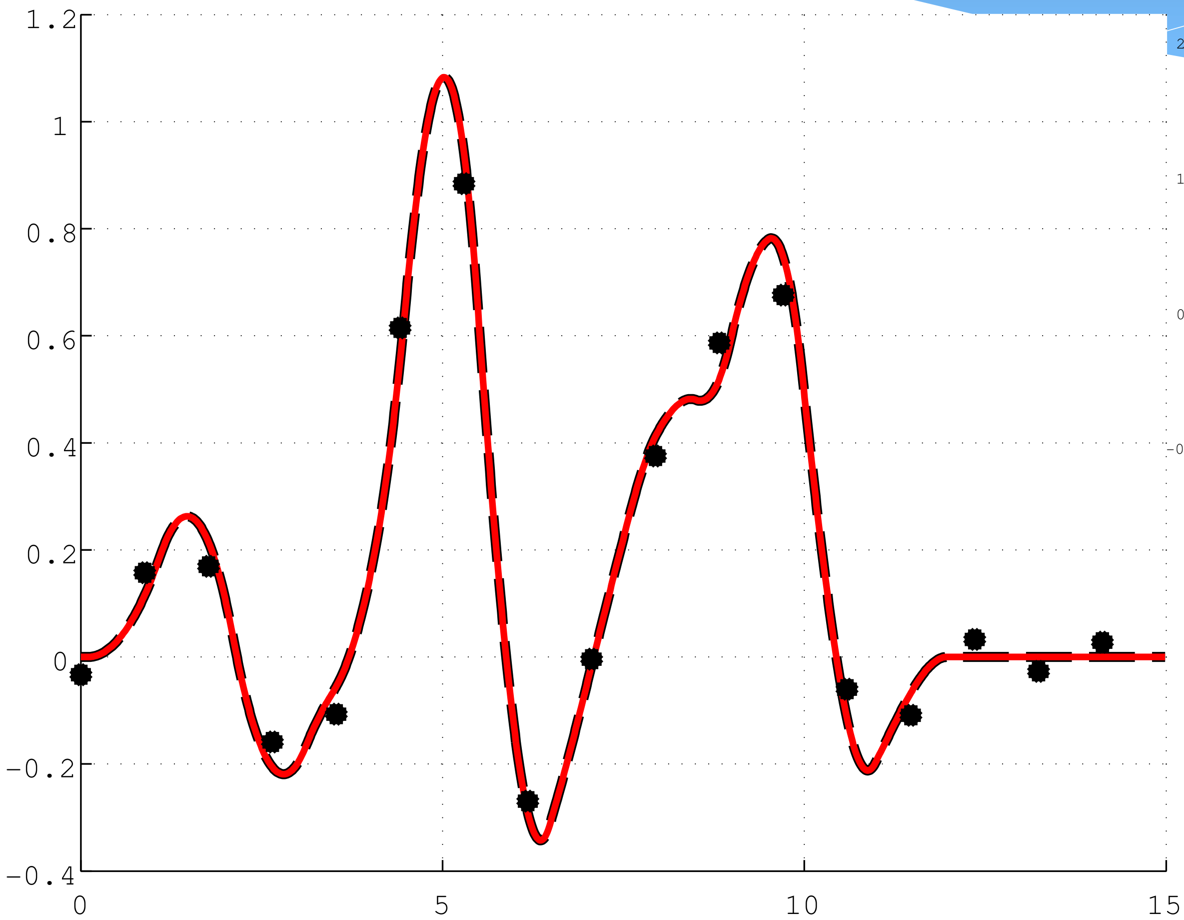
and that

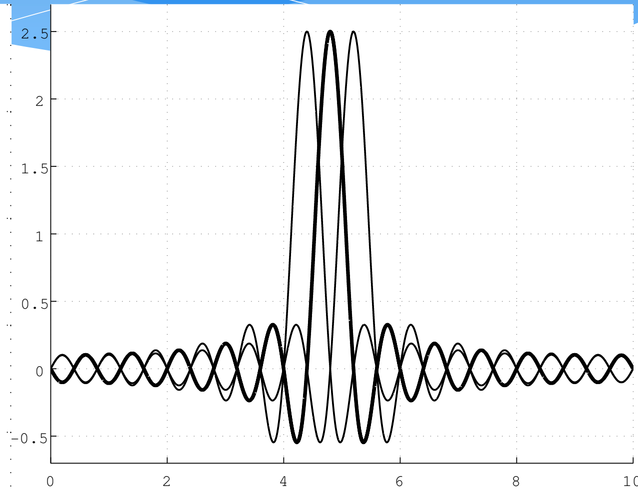
$$N \geq 2P + 1$$

with $P = \lfloor B\tau/2 \rfloor$. If $s(t)$ satisfies $\hat{s}(2p\pi/\tau) \neq 0$ for $p = -P \sim P$, then the samples $\{d_n\}_{n=0}^{N-1}$ using the sinc kernel are a sufficient characterization of the τ-periodic stream of derivative of general pulses.

Derivative of B-Spline of 2^{nd} Deg.

(Hirabayashi, 2012)





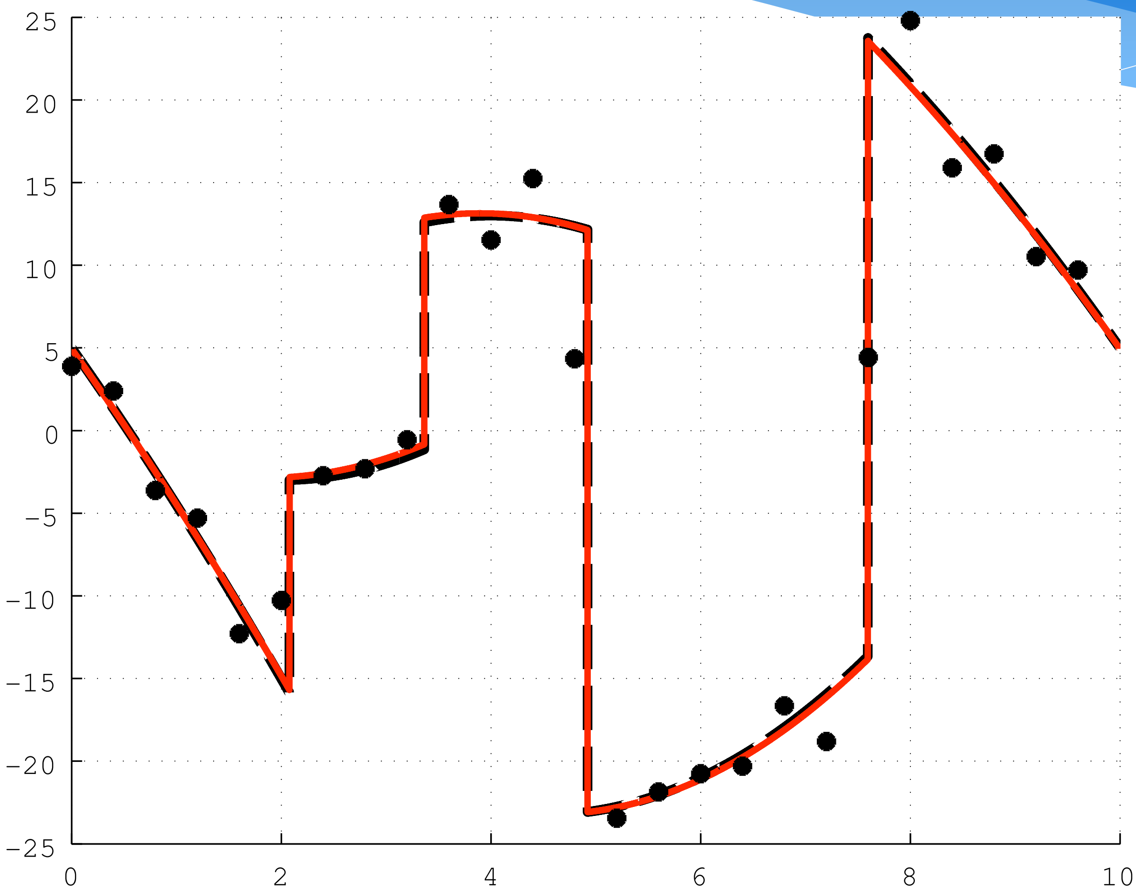
s(t):Quad. Bsplin

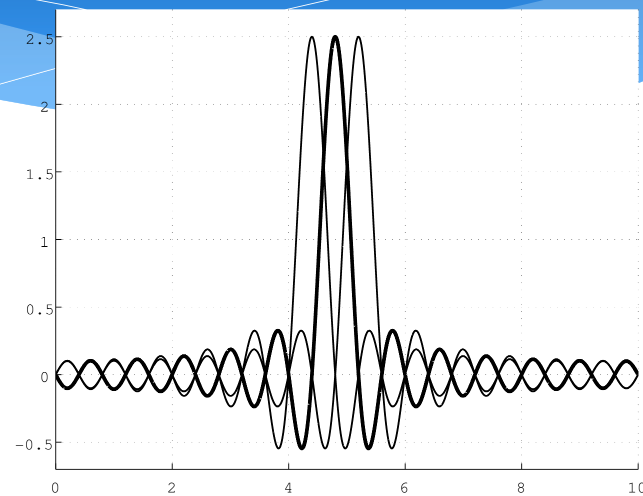
Sinc sampling

K = 4

Periodic Piecewise Polynomial

(Hirabayashi, 2012)



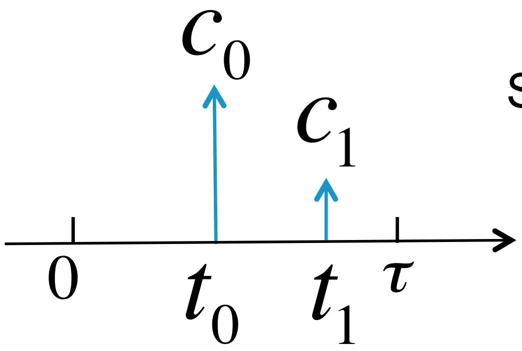


Degree 2 (R=3)
Sinc Sampling
K = 4

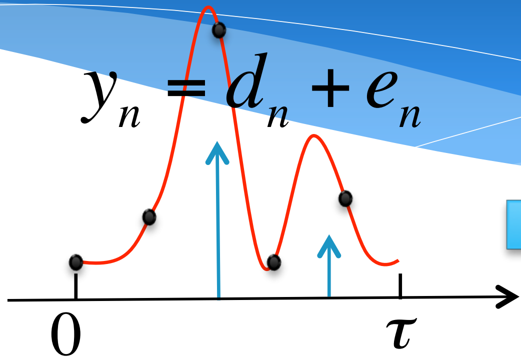
Outline

* Introduction of new class of signals
  * As an extension of bandlimited signals
* Sampling and Reconstruction
  * Noiseless case
  * Noisy case
* Application
  * Compression of ECG signals
  * Line-edge extraction

In Noisy Case



Sampling



DFT

$$\hat{y}_{-2}, \hat{y}_{-1}, \cdots, \hat{y}_2$$

$$\begin{cases} \hat{y}_0 + a_1\hat{y}_{-1} + a_2\hat{y}_{-2} \neq 0 \\ \hat{y}_1 + a_1\hat{y}_0 + a_2\hat{y}_{-1} \neq 0 \\ \hat{y}_2 + a_1\hat{y}_1 + a_2\hat{y}_0 \neq 0 \end{cases}$$

$$\Bigg\| \overbrace{\begin{pmatrix} \hat{y}_0 & \hat{y}_{-1} & \hat{y}_{-2} \\ \hat{y}_1 & \hat{y}_0 & \hat{y}_{-1} \\ \hat{y}_2 & \hat{y}_1 & \hat{y}_0 \end{pmatrix}}^{T} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \Bigg\|_2^2$$

minimization

SVD

Cadzow Denoizing if insufficient

Cadzow Denoizing

$$y_n = d_n + e_n \xrightarrow{\text{DFT}} \hat{y}_p \longrightarrow T$$

$$T = USV^T$$

$$S = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

$$T' = US'V^T$$

$$S' = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Deletion

$$Toeplitz(T')$$

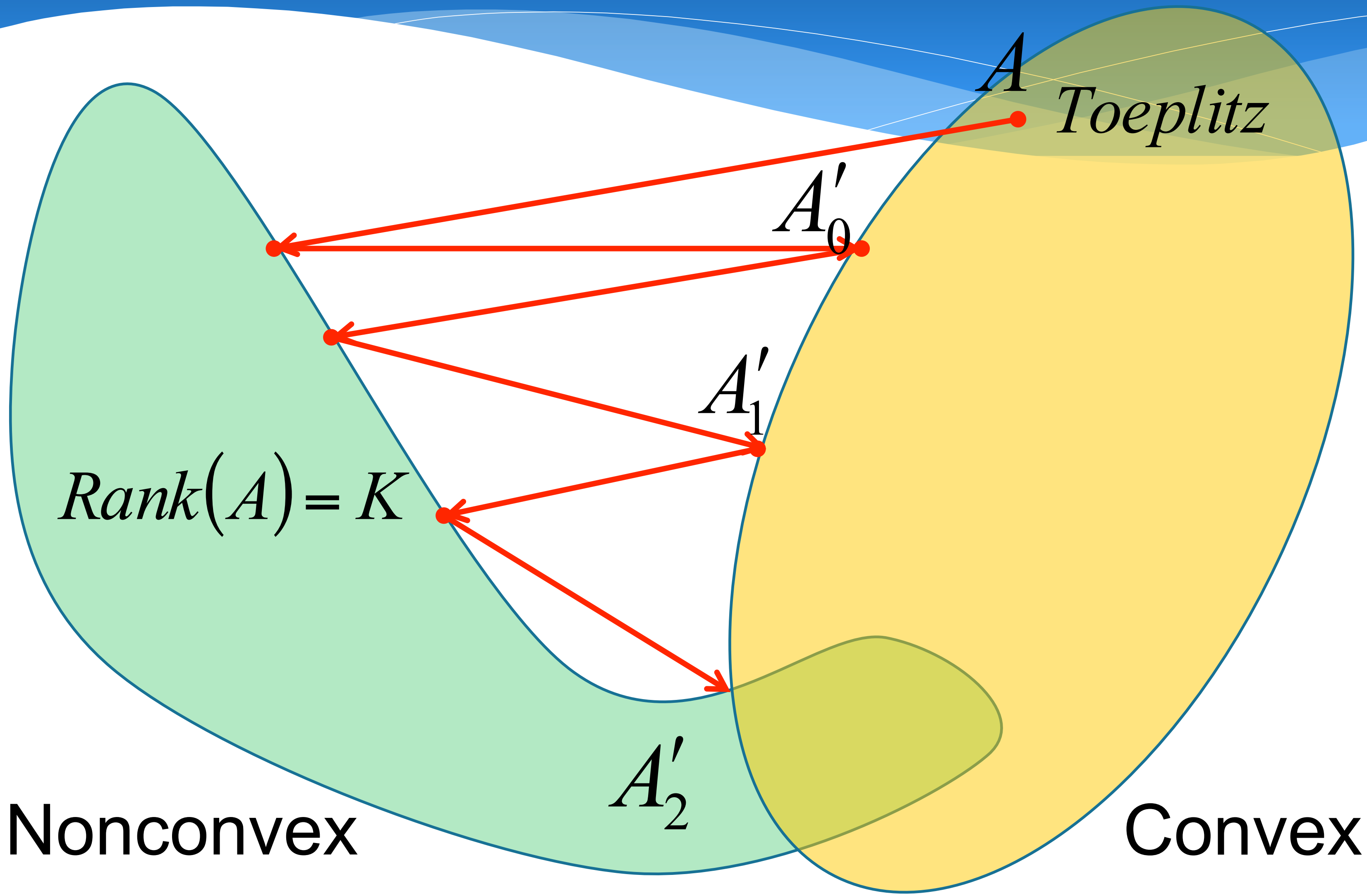
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} t_1 & t_2 & c \\ t_0 & t_1 & t_2 \\ g & t_0 & t_1 \end{pmatrix}$$

$$t_0 = (d + h)/2$$

$$t_1 = (a + e + i)/3$$

$$t_2 = (b + f)/2$$

Cadzow Denoizing



Toward Maximum Likelihood Estimation

$$\hat{\mathbf{d}} = F\mathbf{d} \quad \Longleftrightarrow \quad \mathbf{d} = F^{-1}\hat{\mathbf{d}}$$

$$F = \begin{pmatrix} 1 & e^{i2P\pi/N} & \dots & e^{i2P(N-1)\pi/N} \\ 1 & e^{i2(P-1)\pi/N} & \dots & e^{i2(P-1)(N-1)\pi/N} \\ \dots & \dots & \dots & \dots \\ 1 & e^{-i2P\pi/N} & \dots & e^{-i2P(N-1)\pi/N} \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} d_0 & d_1 & \dots & d_{N-1} \end{pmatrix}^T$$

$$\hat{\mathbf{d}} = \begin{pmatrix} \hat{d}_{-P} & \hat{d}_{-P+1} & \dots & \hat{d}_P \end{pmatrix}^T$$

Toward Maximum Likelihood Estimation (cnt'd)

$$\hat{d}_p = \frac{1}{\tau}\int_0^{\tau} s(t)e^{-i2p\pi t/\tau}\,dt = \frac{1}{\tau}\sum_{k=0}^{K-1} c_k u_k^p$$

$$\hat{\mathbf{d}} = U_t\mathbf{c} \qquad u_k = e^{-i2\pi t_k/\tau}$$

$$\hat{\mathbf{d}} = \begin{pmatrix} \hat{d}_{-P} \\ \hat{d}_{-P+1} \\ \vdots \\ \hat{d}_P \end{pmatrix} U_t = \begin{pmatrix} u_0^{-P} & u_1^{-P} & \cdots & u_{K-1}^{-P} \\ u_0^{-P+1} & u_1^{-P+1} & \cdots & u_{K-1}^{-P+1} \\ \cdots & \cdots & \cdots & \cdots \\ u_0^{P} & u_1^{P} & \cdots & u_{K-1}^{P} \end{pmatrix} \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{pmatrix}$$

Log-Likelihood Function

$$\mathbf{y} = \mathbf{d} + \mathbf{e}$$

↓

$$\mathbf{e} = \mathbf{y} - \mathbf{d} = \mathbf{y} - F^{-1}U_t\mathbf{c}$$

↓

$$p(\mathbf{e}) = p(\mathbf{y} - F^{-1}U_t\mathbf{c})$$

↓

$$l(\mathbf{t},\mathbf{c}) = \log p(\mathbf{y} - F^{-1}U_t\mathbf{c})$$

$$\mathbf{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{pmatrix}$$

Gaussian Distribution

$$l(\mathbf{t},\mathbf{c}) = -\frac{\left\|\mathbf{y} - F^{-1}U_t\mathbf{c}\right\|^2}{2\sigma^2} + \mathrm{Constant}$$

↓

Minimization of $\left\|\mathbf{y} - F^{-1}U_t\mathbf{c}\right\|^2$

↓ *F*: unitary

Minimization of $f_0(\mathbf{t},\mathbf{c}) = \left\|\hat{\mathbf{y}} - U_t\mathbf{c}\right\|^2$

$$\hat{\mathbf{y}} = F\mathbf{y}$$

Reduction of Parameters

For a fixed t,

$$f_0(\mathbf{t},\mathbf{c}) = \left\| \hat{\mathbf{y}} - U_t \mathbf{c} \right\|^2$$

is minimized by

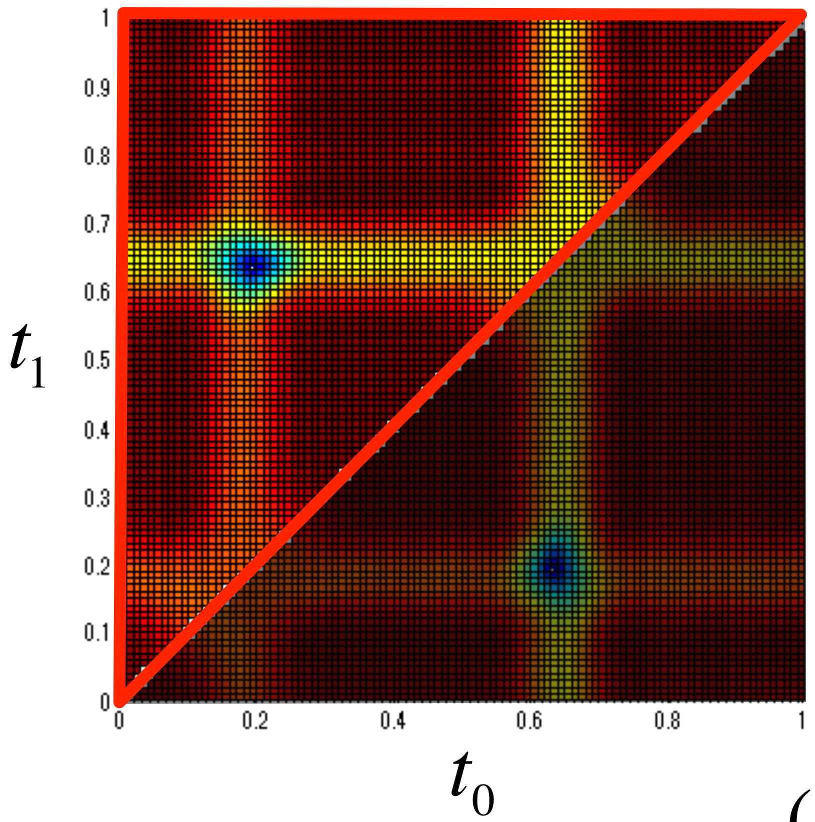
$$\mathbf{c}(\mathbf{t}) = U_t^+ \hat{\mathbf{y}}.$$

Hence, minimizer is obtained by

$$f_0(\mathbf{t},\mathbf{c}(\mathbf{t})) = \left\| \hat{\mathbf{y}} - U_t U_t^+ \hat{\mathbf{y}} \right\|^2 .$$

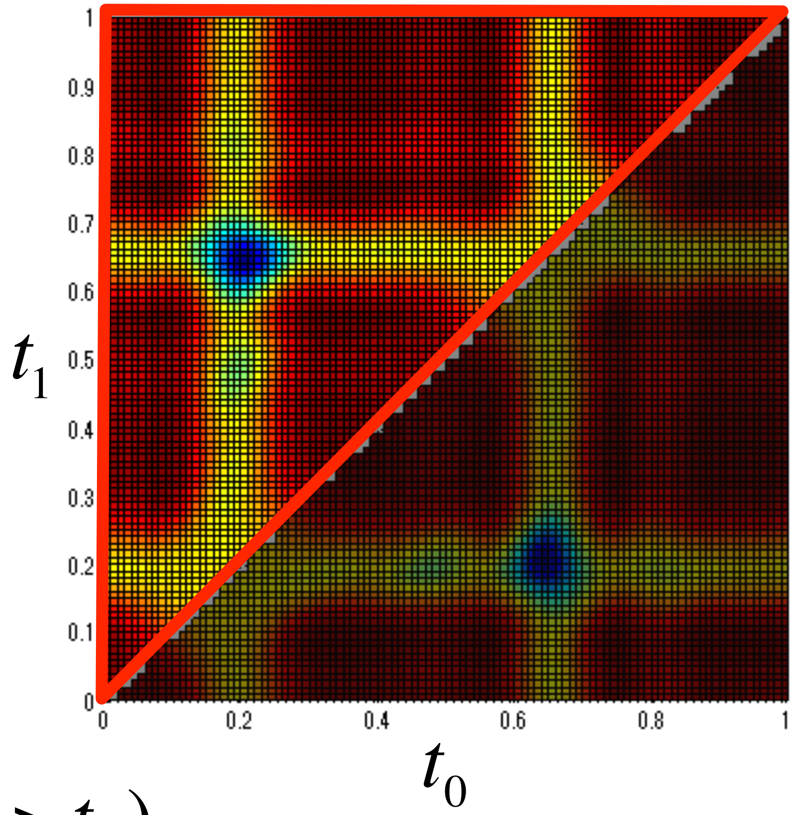
Values of Likelihood Function

Noiseless case



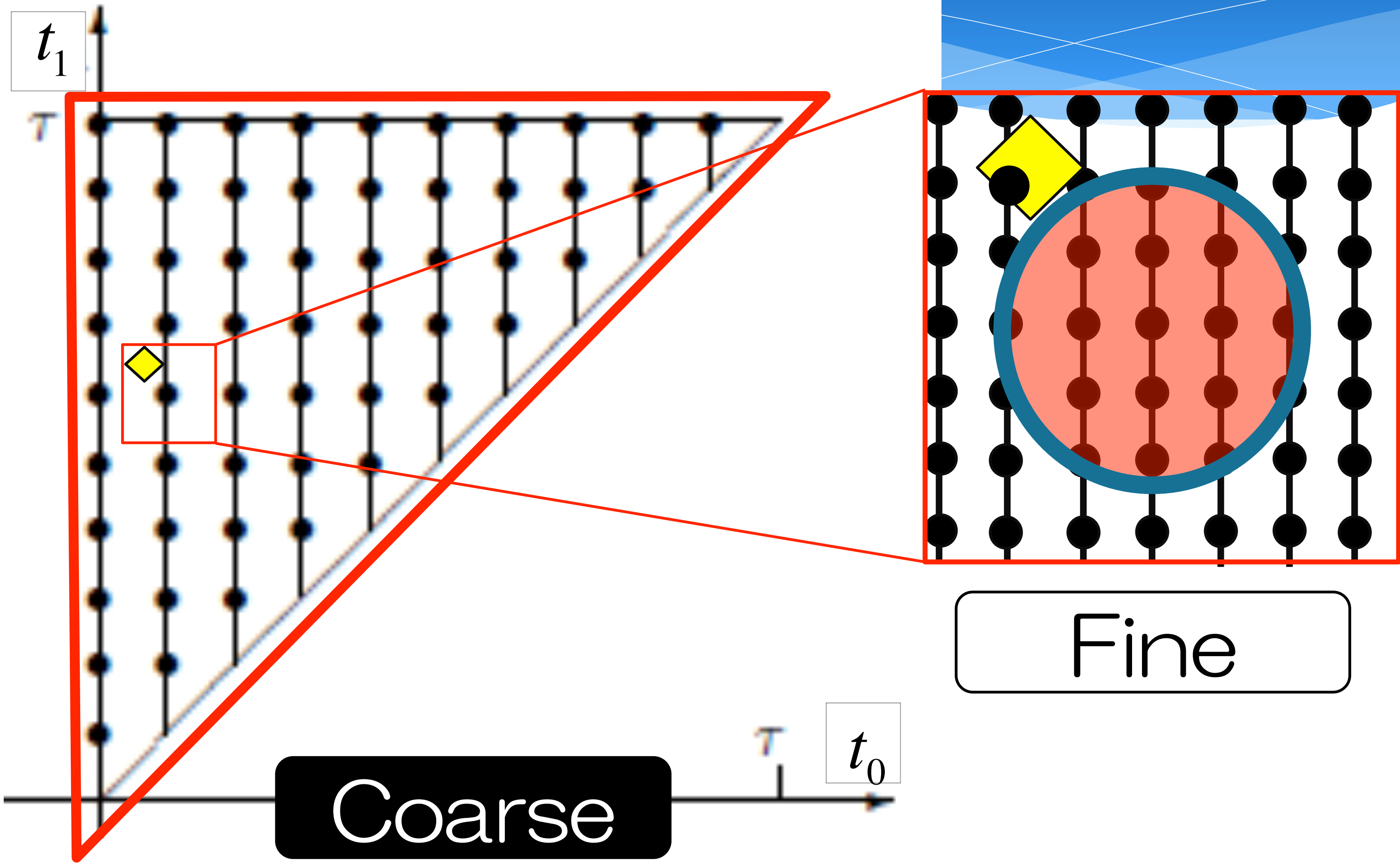
Noisy case

(PSNR=0dB)

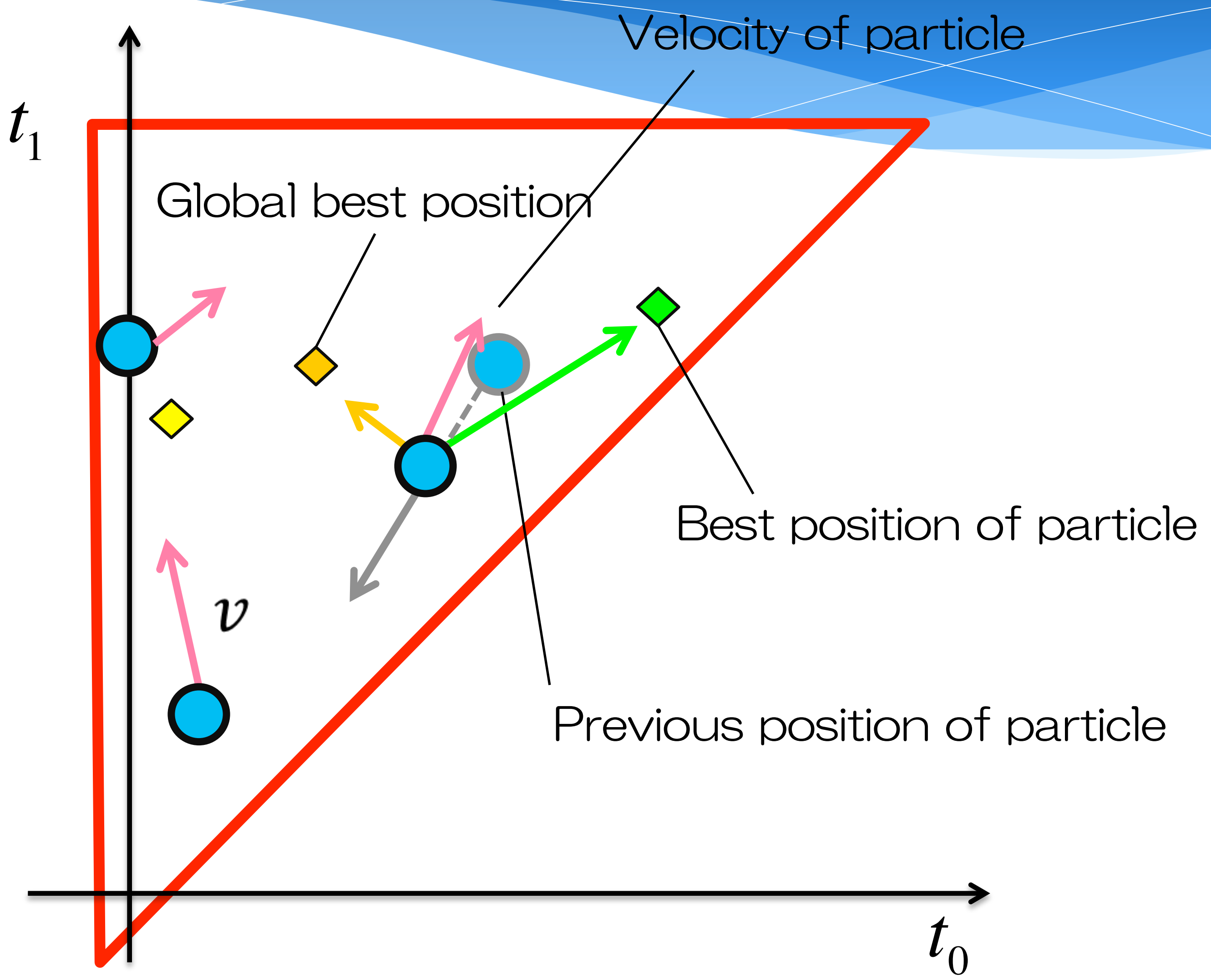


$(t_1 > t_0)$

Coarse to Fine Search

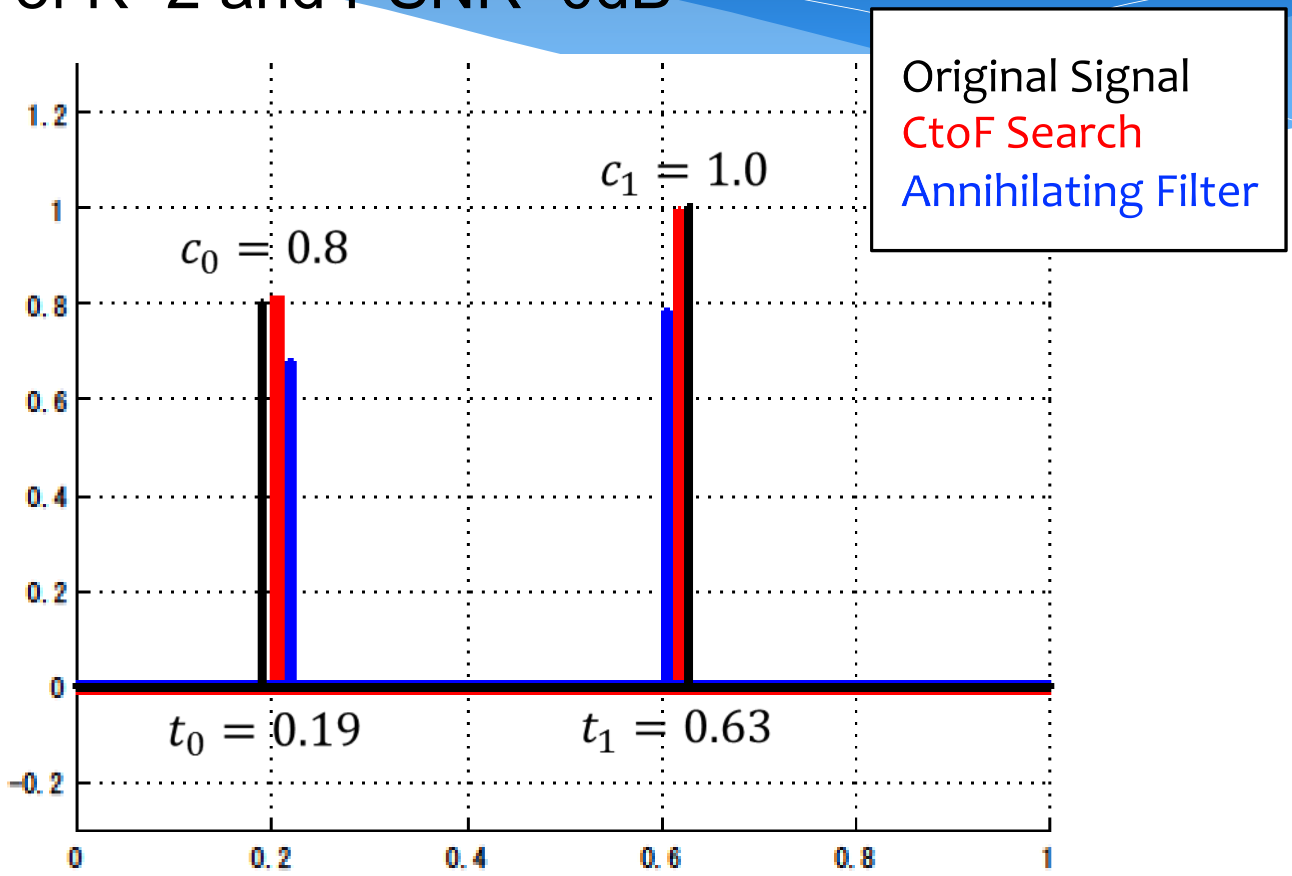


Particle Swarm Optimization



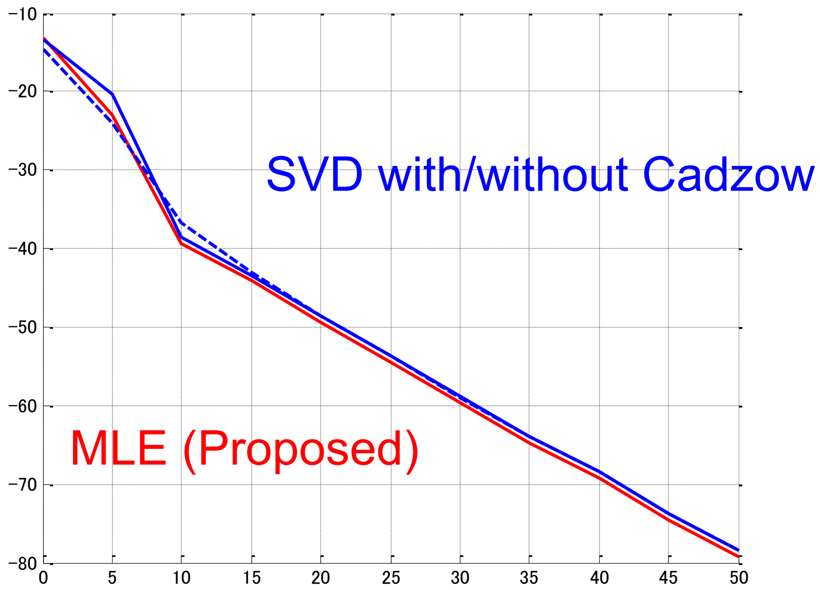
Ex) Reconstruction Result

In case of K=2 and PSNR=0dB



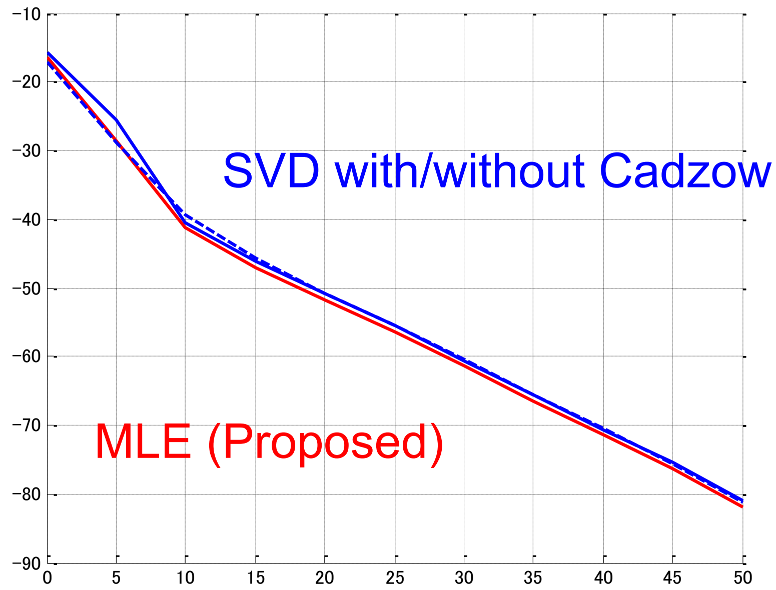
Mean Squared Error for tk

t0



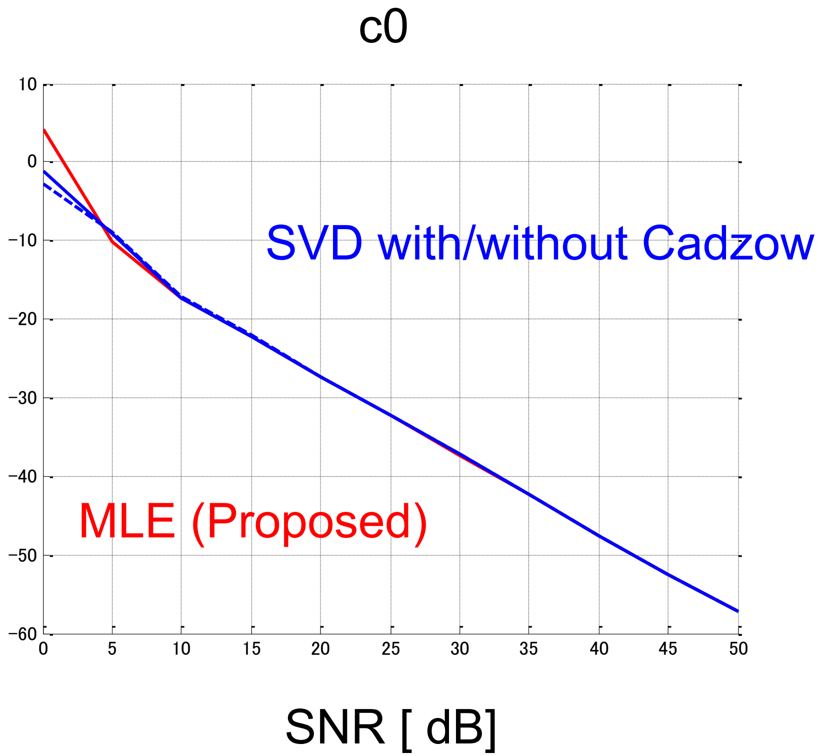
SNR [dB]

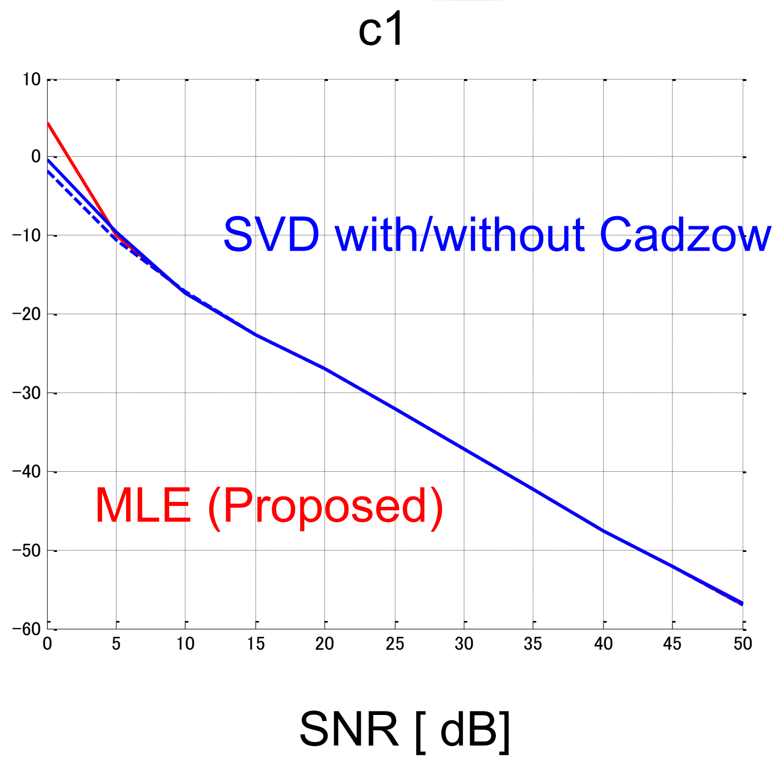
t1



SNR [dB]

Mean Squared Error for ck





Computational Cost